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A NOTE ON AN IMPORTANT THEOREM ON NORMAL DIVISION ALGEBRAS

BY A. A. ALBERT

Dickson has shown that every normal division algebra A of order n^2 over a non-modular field F has rank n, in fact that its rank function $R(\omega; \xi_1, \cdots, \xi_m)$, $m = n^2$, is the characteristic equation of the general element $x = \sum \xi_i u_i$ of A in the expression of A as an algebra of n-rowed square matrices with scalar elements, an equation irreducible in $F(\xi_1, \cdots, \xi_m)$. He has also stated* a theorem upon which a great portion of the research in the theory of division algebras is based.

THEOREM. *Every normal division algebra A of order n^2 over a non-modular field F contains a quantity q whose minimum equation with respect to F has degree n.*

The proof given by Professor Dickson of the above theorem was an immediate application of *Hilbert's irreducibility theorem*, a theorem which has been shown true only for algebraic fields $R(x)$ over R, the field of all rational numbers. It is easy to give examples demonstrating that the property of Hilbert's theorem is not true for the fields of all complex numbers and all real numbers. A less obvious example is the case where F is the field of *all* numbers obtained in a finite number of steps from all rational numbers by the operations of addition, subtraction, multiplication, division except by zero, and extraction of *square* roots. For this case the equation $f(x, t) \equiv x^2 - t = 0$ is evidently irreducible in $F(t)$ when t is an indeterminate. But F is a field containing the square root of any number of F so that the above equation is reducible in F for every value of t in F and the property of Hilbert's theorem does not hold for F.

We shall give a simple algebraic proof of our normal division algebra theorem. The rank function of A is the characteristic equation of x in its expression as an n-rowed square matrix. By a well known theorem on matrices† the characteristic

* *Algebren und ihre Zahlentheorie*, pp. 263–4.

† Ibid., p. 20.

equation of a matrix has the same roots as its minimum equation and, when the latter is irreducible, the former is an exact power of the latter. But for all values of the ξ_i in F the quantities $x=\sum\xi_i u_i$ are in a division algebra and have irreducible minimum equation. Hence $R(\omega;\ \xi_1, \cdots, \xi_m)=0$ is either irreducible in F when the ξ_i take on values in F or is a power of an irreducible equation and is irreducible when it has no multiple roots. But the discriminant $D(\xi_1, \cdots, \xi_m)$ of $R(\omega;\ \xi_i)$ is not identically zero, since $R(\omega;\ \xi_i)$ is irreducible in $F(\xi_1, \cdots, \xi_m)$. Hence* there exists an infinity of values of the ξ_i in F for which $D\neq 0$ and $R=0$, of degree n, is the minimum equation of the corresponding quantities x.

The proof of Hilbert's theorem is non-algebraic and even for fields of algebraic numbers it would be desirable to have an algebraic proof of our important theorem on normal division algebras. The above furnishes such a proof.†

COLUMBIA UNIVERSITY

A COMMUTATION RULE IN QUANTUM MECHANICS

BY EUGENE FEENBERG

In a recent paper N. H. McCoy‡ has developed general commutation rules for the algebra of the quantum mechanics of Born, Heisenberg and Jordan. It is the purpose of this note to point out a commutation rule which in part is implicit in McCoy's work.

The fundamental equation of quantum mechanics from which the algebra is developed is

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* See Fricke, *Algebra*, vol. I, p. 96, for a rational proof of this result which holds for any non-modular field F.

† The author wishes to take this opportunity to announce a correction of the results of his two papers in this Bulletin, vol. 35 (1929), pp. 335–338, and in the Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 372–376, respectively. In both of these papers the Hilbert theorem was used and the results of these papers are correct only for fields for which a *Hilbert irreducibility theorem* is provable. In the statement of Hilbert's theorem in the paper in this Bulletin, the reading should be "K any algebraic field over R, the field of all rational numbers," instead of "K any infinite field."

‡ *Algebra of quantum mechanics*, Transactions of this Society, vol. 31 (1929), pp. 793–806.

$$pq - qp = cI, \tag{1}$$

in which the elements p, q are interpreted as matrices or as non-commutative operators which multiply associatively. I is the identity and c a number, real or complex, but not zero. Unique inverse elements are assumed to exist and are written q^{-1}, p^{-1}. This assumption is not fulfilled for all elements p, q satisfying equation (1), but by restricting exponents to positive values the results obtained are valid independent of the interpretation placed on p, q.

The result to be proved may be stated as follows:

$$(p^iq^i)^m(p^jq^j)^n(q^kp^k)^r(q^lp^l)^s = (q^lp^l)^s(q^kp^k)^r(p^jq^j)^n(p^iq^i)^m, \tag{2}$$
$$i,j,k,l,m,n,r,s = 0, \pm 1, \pm 2, \cdots.$$

By specializing the exponents it is seen that any pair of the terms $(p^iq^i)^m$, $(p^jq^j)^n$, $(q^kp^k)^r$, $(q^lp^l)^s$ commute.

PROOF. Since $pq = qp + cI$, we have

$$(pq)(qp) = (qp)(pq), \quad (p^{-1}q^{-1})(q^{-1}p^{-1}) = (q^{-1}p^{-1})(p^{-1}q^{-1}). \tag{3}$$

Let us set $f = pqp^{-1}q^{-1}$. Then

$$fq^{-1}p^{-1} = (pq)(p^{-1}q^{-1})(q^{-1}p^{-1}) = (pq)(q^{-1}p^{-1})(p^{-1}q^{-1}) = p^{-1}q^{-1},$$

whence

$$f = (p^{-1}q^{-1})\ (q^{-1}p^{-1})^{-1} = (p^{-1}q^{-1})\ (pq).$$

It follows that

$$(pq)(p^{-1}q^{-1}) = (p^{-1}q^{-1})(pq), \quad (qp)(q^{-1}p^{-1}) = (q^{-1}p^{-1})(qp). \tag{4}$$

If G and H are functions of p, q commuting with pq, qp, $q^{-1}p^{-1}$, $p^{-1}q^{-1}$, GH commutes with pq, qp, $q^{-1}p^{-1}$, $p^{-1}q^{-1}$. Since $(pq)^{-n} = (q^{-1}p^{-1})^n$ and $(p^{-1}q^{-1})^{-n} = (qp)^n$ we find from (3) and (4) that

$$(pq)^m(qp)^n = (qp)^n(pq)^m, \qquad (n, m = 0, \pm 1, \pm 2, \cdots). \tag{5}$$

To complete the proof we show that

$$p^nq^n = \sum_{i=0}^{n} a_{in}(pq)^i, \quad q^np^n = \sum_{i=0}^{n} b_{in}(qp)^i, \tag{6}$$

where $n = 0, 1, 2, \cdots$, and the coefficients a_{in}, b_{in} are numbers.

$$p^2q^2 = p(pq)q = p(qp + cI)q = (pq + cI)pq.$$

It follows by induction that

$p^nq^n=(pq+(n-1)cI)p^{n-1}q^{n-1}$, $\quad q^np^n=(qp-(n-1)cI)q^{n-1}p^{n-1}$,

$n=0, 1, 2, \cdots$, from which (6) is readily proved. Then, by (5), we have

$$
\begin{aligned}
&(p^nq^n)(p^mq^m) = (p^mq^m)(p^nq^n),\\
(7)\qquad &(p^nq^n)(q^mp^m) = (q^mp^m)(p^nq^n),\\
&(q^np^n)(q^mp^m) = (q^mp^m)(q^np^n), \qquad (m, n = 0, 1, 2, \cdots).
\end{aligned}
$$

From (7) we obtain

$$
\begin{aligned}
(q^{-m}p^{-m})(q^{-n}p^{-n}) &= (q^{-n}p^{-n})(q^{-m}p^{-m}),\\
(q^{-m}p^{-m})(p^nq^n) &= (p^nq^n)(q^{-m}p^{-m}),
\end{aligned}
$$

and four similar equations. Therefore (7) is true for m, $n=0$, ±1, ±2, $\cdots$, and since $(p^iq^i)^{-n}=(q^{-i}p^{-i})^n$ we have the result which was to be proved:

$$
\begin{aligned}
(p^iq^i)^m(p^jq^j)^n &= (p^jq^j)^n(p^iq^i)^m,\\
(p^iq^i)^m(q^jp^j)^n &= (q^jp^j)^n(p^iq^i)^m,\\
(q^ip^i)^m(q^jp^j)^n &= (q^jp^j)^n(q^ip^i)^m, \qquad (i, j, m, n = 0, \pm 1, \pm 2, \cdots).
\end{aligned}
$$

By specializing a general identity McCoy finds

$$q^mp^{m'}q^np^{n'}=q^{n'}p^nq^{m'}p^m, \qquad m+n=m'+n'.$$

To illustrate the use of the theorem we verify this identity:

$$
\begin{aligned}
q^mp^{m'}q^np^{n'} &= (q^mp^m)(p^{m'-m}q^{m'-m})(q^{n'}p^{n'})\\
&= (q^{n'}p^{n'})(p^{m'-m}q^{m'-m})(q^mp^m) = q^{n'}p^nq^{m'}p^m.
\end{aligned}
$$

If inverses do not exist and $m'<m$

$$
\begin{aligned}
q^mp^{m'}q^np^{n'} &= q^{m-m'}(q^{m'}p^{m'})(q^np^n)p^{n'-n}\\
&= q^{m-m'}(q^np^n)(q^{m'}p^{m'})p^{n'-n} = q^{n'}p^nq^{m'}p^m.
\end{aligned}
$$

HARVARD UNIVERSITY

NOTE ON THE EQUIVALENCE OF CERTAIN PROPERTIES OF ABSTRACT SETS*

BY R. G. PUTNAM

Sierpinski† has shown that, in classes (S) (that is, classes (L) in which every derived set is closed), the following three properties are equivalent:

(1) Every uncountable set contains at least one point of condensation.

(2) Every clairsemé set is at most countable.

(3) Every infinite well ordered descending sequence of closed distinct sets is countable.

The equivalence of these three properties is shown by proving that property (1) implies (2), that (2) implies (3) and that (3) implies (1). The condition that every derived set be closed is used only in showing that (3) implies (1).

Sierpinski also proves in the same paper that, in classes (S), properties (4) and (5) which follow are equivalent:

(4) Every infinite well ordered ascending sequence of closed distinct sets is countable.

(5) Every set E of the class contains a countable subset D dense in E (that is, $E \subset D+D'$).

The equivalence of these two properties is obtained by proving that (4) implies (5) and that (5) implies (4). The condition that every derived set be closed is used only in the proof that (4) implies (5).

Properties (3) and (4) are shown by means of examples to be independent of each other.

It is the purpose of this note to point out that the above equivalences hold without change for classes (V) in which every derived set is closed and from this result theorems concerning classes (V) are obtained. The method of showing the equivalences in classes (V) is the same as that used for classes (L), definitions of limit point, point of condensation, etc., in classes

* Presented to the Society, December 29, 1929.

† *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fundamenta Mathematicae, vol. 2, p. 179.

(L) being replaced by those which apply to classes (V). For example, in classes (L) a point of condensation of a set E is a limit point of E which remains a limit point of every set obtained by removing a countable set of points from E; in classes (V) a point of condensation of a set E is a point in every neighborhood of which there is an uncountable set of points of E.

From the proof that property (1) implies property (2) we have, for all classes (V), the following theorem.

THEOREM. *In a class (V) condensed in itself,* every uncountable set E is the sum of two disjoined sets, one of which is dense in itself (and not empty) and the other clairsemé and at most countable.*

Since property (1) implies property (2) which in turn implies Property (3) in any class (V), we may state the theorem:

THEOREM. *In every class (V) condensed in itself, every infinite descending sequence of distinct closed sets is countable.*

This theorem has been shown by Fréchet† to hold for perfectly separable sets.‡

From the fact that property 5 implies property 4 we obtain the following theorem.

THEOREM. *Every infinite well ordered ascending sequence of closed distinct sets of a separable class (V) is countable.*

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* A set is condensed in itself if every uncountable subset contains at least one point of condensation.

† Fundamenta Mathematicae, vol. 10, p. 334.

‡ A set is perfectly separable if its limit points can be defined in terms of a countable family of neighborhoods. A perfectly separable set is condensed in itself. See Fréchet, *Les Espaces Abstraits*, p. 233.

ON A COMPLETE CHARACTERIZATION OF THE SET OF POINTS OF UNBOUNDED GRADE OF AN ARBITRARY SURFACE*

BY HENRY BLUMBERG

Let $z=f(x, y)$ be an arbitrary surface S, in the sense that $f(x, y)$ is an arbitrary one-valued real function of the real variables x and y. By the *grade* of a segment joining two points A and B, we understand the absolute value of the tangent of the angle which AB makes with the xy-plane. The point $A=(\xi, \eta, \zeta)$ of the surface S is said to be of *bounded grade*—or S is said to be of bounded grade at the point (ξ, η)—if the grade of AB is bounded for all $B=(x, y, z)$ of S at a sufficiently small "horizontal" distance $[(x-\xi)^2+(y-\eta)^2]^{1/2}$ from A. If A does not satisfy this condition, S is said to be of unbounded grade at (ξ, η). It is the object of the present paper to prove the following theorem, which identifies the aggregate—for the totality of arbitrary surfaces—of sets of points of unbounded grade with the aggregate of sets of type G_δ.†

THEOREM.‡ *The set of points (x, y) at which an arbitrary surface $z=f(x, y)$ is of unbounded grade is a G_δ. Conversely if a G_δ is given, there exists a surface $z=f(x, y)$ such that this G_δ is identical with the set of points (x, y) where the surface is of unbounded grade.*

PROOF. If $P=(x, y)$ is a point at which the given surface S, represented by $z=f(x, y)$, is of unbounded grade, we may, for every positive integer n, find a point $P_n=(x_n, y_n)$ such that the distance $d(PP_n)$ between P and P_n is less than $1/n$, and $g(f, PP_n)>n$, understanding by $g(f, PP_n)$ the grade of the segment joining the points of S corresponding to P and P_n. Enclose P and P_n in a circle $C_P^{(n)}$, regarded as made up only of

* Presented to the Society, April 16, 1927.

† A G_δ is the product of $\aleph_0$ open sets. The notation is due to Hausdorff.

‡ The direct part of this theorem is stated by W. H. Young for the case of a function of one variable; see Arkiv för Matematik, Astronomi och Fysik, vol. 1 (1903).

interior points, of diameter $<2d(PP_n)$. Let G_n equal the sum of the $C_P^{(n)}$ for n fixed and P ranging over the set U of points at which S is of unbounded grade; and let $T=\prod_1^\infty G_n$. We show that $T\equiv U$. Of course, since every point of U is in G_n for every n, we have $U\subseteqq T$. Suppose Q is in $C_P^{(n)}$. Then either $|f(Q)-f(P)|$ or $|f(Q)-f(P_n)|$, where $f(P)=f(x, y)$, with similar meaning for $f(Q)$ and $f(P_n)$, is not less than $\frac{1}{2}|f(P)-f(P_n)|$. Since $d(QP)$ and $d(QP_n)$ are both less than $2d(PP_n)$, we conclude that either $g(f, QP)$ or $g(f, QP_n)>g(f, PP_n)/4>n/4$. It follows that if Q is in some $C_P^{(n)}$ for every n, then S is of unbounded grade at Q. Hence $T\subseteqq U$, and therefore $T\equiv U$.

To prove the converse part of the theorem, we suppose that $\prod_1^\infty G_n$ is a given product of open sets G_n lying in the xy-plane. Let $G^{(n)}=\prod_1^n G_n$, and $F_n=$ the complement of $G^{(n)}$ with respect to the xy-plane. In terms of these F_n, we shall define the surfaces $z=f_n(x, y)$; and $z=f(x, y)=\sum_1^\infty f_n(x, y)$ will be the required surface having bounded grade at the points of $\sum_1^\infty F_n$ and unbounded grade at the points of $\prod_1^\infty G_n$.

To this end, we suppose that $T_n=\{Q^{(n)}\}$ is, for every positive integer n, a system of non-overlapping squares $Q^{(n)}$ lying in $G^{(n)}$, such that every point of $G^{(n)}$ is in the interior or on the boundary of at least one $Q^{(n)}$ of T_n. We define $f_n(x, y)$ as 0 at the points of F_n and at the boundary points of the $Q^{(n)}$; if $P=(x, y)$ is an interior point of the square $Q^{(n)}$, we set

$$f_n(P)=f_n(x, y)=\rho_n d_P^{(n)},$$

where $d_P^{(n)}$ is the distance from P to the boundary of $Q^{(n)}$, and ρ_n is a number, depending on n but not on the varying $Q^{(n)}$ of T_n, and subject to certain relations to be stated presently. We suppose furthermore that T_{n+1} is a "subdivision" of $T_n\{n=1, 2, \cdots\}$ in the sense that every $Q^{(n+1)}$ of T_{n+1} lies in just one $Q^{(n)}$ of T_n. Let $2q^{(n)}$, which may vary, as $Q^{(n)}$, n fixed, varies, be the length of side of $Q^{(n)}$. Then, as we may, we select the ρ_n and $q^{(n)}$ in such a way that

(a) $\rho_n/\sqrt{2} > M_{n-1}+n (n\geqq 2);\ \rho_1=1;$

(b) $\rho_n q^{(n)} < d_n^2/2^n;$

(c) $\rho_n q^{(n)} < q^{(n-1)}.$

Here M_n represents the upper boundary, which is evidently

finite, of the grade of a segment with end points on the surface

$$z = s_n(x,\ y) = \sum_1^n f_\nu(x,\ y),$$

for example, $M_1 = 1$; and d_n, which depends, for n fixed, on $Q^{(n)}$, is the minimum distance from the boundary points of $Q^{(n)}$ to F_n; moreover, inequality (c) is to be understood as demanded only in case the square $Q^{(n)}$ of side $2q^{(n)}$ lies in, or has at least one boundary point in common with the square $Q^{(n-1)}$ of side $2q^{(n-1)}$. We now set $f(x,\ y) = \sum_1^\infty f_n(x,\ y)$, and observe that f exists. For, by the definition of f_n and inequality (b), if P is an interior point of some $Q^{(n)}$ of T_n, then

$$f_n(P) = \rho_n d_P^{(n)} \leqq \rho_n q^{(n)} < d_{nP}^2/2^n,$$

where d_{nP} is the minimum distance from P to F_n. Since d_{nP} does not increase with n, and $f_n(P) = 0$ if P is interior to no $Q^{(n)}$ of T_n, it follows that $\sum f^n(P)$ is convergent. We shall now prove that f has the required properties.

First, let P be a point of F_n. If $\nu > n$, and P' is a point of G_ν lying in the interior of the square $Q^{(\nu)}$ of T_ν, we have, by inequality (b),

$$g(f_\nu,\ PP') < \frac{\rho_\nu q^{(\nu)}}{d(PP')} < \frac{\rho_\nu q^{(\nu)}}{d_\nu} < \frac{d_\nu}{2^\nu} < \frac{d(PP')}{2^\nu}.$$

If $P' \neq P$ is interior to no $Q^{(\nu)}$ of T_ν, $f_\nu(P') = 0$, and since $f_\nu(P) = 0$, we have $g(f_\nu,\ PP') = 0$. It follows, if $r_n(x,\ y) = \sum_{n+1}^\infty f_\nu(x,\ y)$, that $g(r_n,\ PP') < d(PP')$ for every point $P' \neq P$. Since the surface $z = s_n(x,\ y)$ is of bounded grade at every point, and

$$f(x,\ y) = s_n(x,\ y) + r_n(x,\ y),$$

we conclude that the surface $z = f(x,\ y)$ is of bounded grade at every point of every F_n, and therefore of bounded grade at every point of $\sum F_n$.

Now let P be a point of $\prod G_n$, and $Q^{(n)}$ a square of T_n containing P in its interior or on its boundary. Then there is a point P' in $Q^{(n)}$ such that $d(PP') \geqq q^{(n)}/2$ and $g(f_n,\ PP') \geqq \rho_n/\sqrt{2}$. Since $g(s_{n-1},\ PP') \leqq M_{n-1}$, we have, by inequality (a), the relation $g(s_n,\ PP') > n (n \geqq 2)$. Let ν be an integer greater than n. If P and P' both belong to $G^{(\nu)}$ there are two squares of T_ν, possibly identical, the one containing P and the other P'; let

$Q^{(\nu)}$ be the one with the larger (or at least not the smaller) side $q^{(\nu)}$. Then

$$g(f_\nu,\ PP') \leqq \rho_\nu q^{(\nu)}/d(PP') < 2\rho_\nu q^{(\nu)}/q^{(n)}.$$

Moreover $Q^{(\nu)}$ lies in a $Q^{(\nu-1)}$ and this in turn in a $Q^{(\nu-2)}$ and so on down to $Q^{(n+1)}$, which lies in or has a boundary point in common with $Q^{(n)}$. Therefore, in virtue of inequality (c),

$$g(f_\nu,\ PP') < \frac{2q^{(\nu-1)}}{q^{(n)}} = \frac{2q^{(\nu-1)}}{q^{(\nu-2)}} \cdots \frac{q^{(n+1)}}{q^{(n)}}$$

$$< 2\frac{1}{\rho_{\nu-1}} \cdots \frac{1}{\rho_{n+1}} < \frac{1}{2^{\nu-n-2}},$$

since $\rho_n > 2$ for $n > 1$. The reasoning here implies at first that $\nu > n+1$, but the final inequality is valid for $\nu = n+1$ also, and evidently, too, if either one or both of the points P and P' lie in F_ν. Hence

$$g(r_n,\ PP') < \sum_{\nu=n+1}^{\infty} \frac{1}{2^{\nu-n-2}} = 4.$$

Therefore, for $n \geqq 2$,

$$g(f,\ PP') \geqq g(s_n,\ PP') - g(r_n,\ PP') > n - 4.$$

Since P belongs to $\prod G_n$, a point P' satisfying the last inequality can be found for every positive integer $n \geqq 2$; and since PP', together with $q^{(n)}$ is, by (c), infinitesimal as $n \to \infty$, we conclude that the surface $z = f(x, y)$ is of unbounded grade at P.

REMARK. The proof has been given for a surface $z = f(x, y)$ lying in a euclidean space, but the same reasoning applies to euclidean n-space. In fact, with certain modifications of the argument not hard to discern, our theorem, including the converse part, can be extended to any metric space, assumed, of course, if the theorem is to retain significance, to be without isolated points. The triangle postulate $d(P_1P_2) + d(P_2P_3) \geqq d(P_1P_3)$ for such a space turns out to be an adequate substitute for metric relations in the plane frequently utilized in the proof. For such an abstract space, we should, for example, change the squares to "spheres"; however, to show that $G^{(n)}$ is the sum of non-overlapping spheres $Q^{(n)}$, boundary included, we make use of Zermelo's Theorem on Normal Order.

OHIO STATE UNIVERSITY

ON THE DENSITY OF THE CUT POINTS AND END POINTS OF A CONTINUUM*

BY W. L. AYRES

1. *Introduction.* We consider a connected, compact, metric space M which we refer to as the continuum M. The space M is said to be *locally connected* if for each point p of M and each $\epsilon>0$ there exists a $\delta>0$ and a connected set N such that $S(p, \delta) \subset N \subset S(p, \epsilon)$. A point p is said to be a *cut point* of M if $M-p$ is not connected. A point p is said to be an *end point* of M if for each $\epsilon>0$ there exists a neighborhood U_p such that $U_p \subset S(p, \epsilon)$ and $B(U_p)=\overline{U}_p-U_p$ is a single point.† From this definition it is seen that every end point is a limit point of the cut points of M. Hence whenever the end points are dense in M, the cut points are also dense. This relation is not true conversely, but a study of some examples leads one to the conclusion that there exist some fundamental relations between the density of the cut points and the density of the end points. In this note we shall investigate some of these relations.

2. *Notation.* Let K and E denote respectively the set of all cut points and end points of M. Let K^2 denote the set of all cut points which are of Urysohn-Menger order 2 in M and let ${}^2K$ denote the set of all cut points which are of order >2. Capitals will denote sets of points, lower case letters single points. $S(p, \epsilon)$ denotes the set of all points whose distance from p is less than ϵ. The symbol $\rho(x, y)$ denotes the distance from x to y; $\rho(X, Y)$ denotes the greatest lower bound of the numbers $\rho(x, y)$ where $x \varepsilon X$ and $x \varepsilon Y$. The notations $x \varepsilon X$ and x non-ε X mean "x is a point of the set X" and "x is not a point of X" respectively. The symbol $d(X)$ denotes the diameter of X, that is, the least upper bound of all numbers $\rho(x, y)$ where $x \varepsilon X$ and $y \varepsilon X$.

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* Presented to the Society, August 30, 1929.

† We use end point here in the Urysohn-Menger sense. See P. Urysohn, Comptes Rendus, vol. 175 (1922), pp. 481–483; and K. Menger, Mathematische Annalen, vol. 95 (1925), pp. 277–306. For other senses in which the term has been used, see H. M. Gehman, *Concerning end points of continuous curves and other continua*, Transactions of this Society, vol. 30 (1928), pp. 63–84.

3. THEOREM. *If the end points are dense in $M-K$, the cut points are dense in M.*

Since each end point is a limit point of the cut points, we have $\overline{E} \subset \overline{K}$. By hypothesis, $M-K \subset \overline{E}$. Then $M-K \subset \overline{K}$. As $K \subset \overline{K}$, we have $M=\overline{K}$.

4. THEOREM. *In order that the cut points of a locally connected continuum M be dense in M it is necessary and sufficient that the end points be dense in $M-K$.* $[(\overline{K}=M) \equiv (\overline{E} \supset M-K).]$

The condition is sufficient by the preceding theorem. Now let $p \varepsilon M-K-E$ and choose $\epsilon>0$. There exists just one maximal cyclic set C of M containing p.* By hypothesis there exists a $q_1' \varepsilon K \cdot S(p, \epsilon/2)$. If $q_1' \varepsilon C$, let $q_1=q_1'$ and let H_1 be a component of $M-q_1$ such that $H_1 \cdot C=0$. If q_1' non-ε C, let H_1 be the component of $M-C$ containing q_1' and let $q_1 \varepsilon C \cdot \overline{H}_1$. Evidently $q_1=C \cdot \overline{H}_1$ and $q_1 \neq p$. There exists an integer $m_2>2$ such that $S(p, \epsilon/m_2) \cdot H_1=0$. Let $q_2' \varepsilon K \cdot S(p, \epsilon/m_2)$. If $q_2' \varepsilon C$, let $q_2=q_2'$ and H_2 be a component of $M-q_2$ such that $H_2 \cdot C=0$. If q_2' non-ε C, let H_2 be the component of $M-C$ containing q_2' and let $q_2=C \cdot \overline{H}_2$. We see quite easily that $H_2 \cdot H_1=0$ and $q_2 \neq p$. In general there exists an integer $m_i>m_{i-1}(i>1)$ such that $S(p, \epsilon/m_i) \cdot H_{i-1}=0$. Let $q_i' \varepsilon K \cdot S(p, \epsilon/m_i)$ and we find H_i and q_i from this exactly as above. Again we have $q_i \neq p$ and $H_i \cdot H_j=0$. Then we have an infinite set of distinct components of $M-C$, $H_1, H_2, H_3, \cdots$, such that

$$\overline{H}_i \supset q_i', \qquad \lim_{i \to \infty} q_i' = p, \tag{1}$$

$$\overline{H}_i \cdot C = q_i \neq p. \tag{2}$$

Since M is locally connected it follows from (1) and (2) that

$$\lim_{i \to \infty} q_i = p. \tag{3}$$

Now as $d(H_i)$ approaches zero it follows from (3) that there

* We are using the term *maximal cyclic set* in place of Whyburn's term *maximal cyclic curve*. For definitions and properties of maximal cyclic sets, see G. T. Whyburn, *Cyclicly connected continuous curves*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 31–38. These results are extended to metric space in my paper, *Concerning continuous curves in metric space*, American Journal of Mathematics. vol. 51 (1929), pp. 577–594.

exists an integer n such that $S(p, \epsilon) \supset H_n$. If $\overline{H}_n$ is cyclicly connected, then $\overline{H}_n \cdot K = q_n$. Then no point of H_n is a limit point of K contrary to hypothesis. Then $\overline{H}_n$ is not cyclicly connected and thus contains two nodes of itself.* One of these nodes N does not contain q_n. Either N is an end point of $\overline{H}_n$ or N is a maximal cyclic set of $\overline{H}_n$ containing just one cut point of $\overline{H}_n$. But in this last case N is a maximal cyclic set of M and contains just one point of K. But this is impossible just as above. Hence N is an end point of $\overline{H}_n$ and it is not difficult to see that $N \varepsilon E$. Then for each $\epsilon > 0$ there exists an $N \varepsilon E \cdot S(p, \epsilon)$. Hence $p \varepsilon \overline{E}$.

5. *Example.* We may see quite simply that the condition of the preceding theorem is not necessary without the restriction that M be locally connected. Let M be the curve $y = \sin 1/x$ for $0 < x \leqq 1$ together with the points of the y-axis for which $-1 \leqq y \leqq 1$. Then $\overline{K} = M$ but the points of M on the y-axis belong to $M - K$ and no one of them belongs to $\overline{E}$. In fact E is the single point (1, sin 1).

6. THEOREM. *In order that the end points of a locally connected continuum M be dense in M it is necessary and sufficient that the cut points of order >2 be dense in M.* $[(\overline{E} = M) \equiv (^2\overline{K} = M).]$

The condition is necessary. Since $E \neq 0$, the continuum M is not cyclicly connected. Then if M contains a maximal cyclic set C, each component of $M - C$ has just one limit point in C and thus $C \cdot K \neq 0$. Further $C \cdot K = C \cdot {}^2K$, for if $x \varepsilon C \cdot K$ there exist at least three arcs meeting at x (two arcs on a simple closed curve of C containing x and one arc in a component of $M - C$ whose limit point is x) and so x is of order $\geqq 3$ in M.

Now let $p \varepsilon E$. If p non-ε $^2\overline{K}$, there exists a neighborhood U_p such that $U_p \cdot {}^2K = 0$. As $p \varepsilon E$ there exists a neighborhood V_p such that $\overline{V}_p \subset U_p$ and $B(V_p) = q$, a single point. Let A_1 be an arc of M with end points p and q and let $y \varepsilon A_1 - p - q$. Evidently $A_1 \subset \overline{V}_p$. From the local connectivity there exists a neighborhood $W_y \subset V_p - p$ such that if $z \varepsilon W_y$ then there is an arc from z to y which lies in V_p. As $\overline{E} = M$ let $z \varepsilon E \cdot W_y$. Then $z \varepsilon M - A_1$. Let A_2 be the arc of M joining z and y such

* G. T. Whyburn, *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), pp. 167–194, Theorem 14.

that $V_p \supset A_2$. In the order from z to y let w be the first point of A_1 on A_2. The point $w \neq p$ for $p \varepsilon E$ and thus is not interior to an arc of M. The point w is of order $\geqq 3$ since there are at least three arcs (subarc wp of A_1, subarc wq of A_1, subarc wz of A_2) meeting in w. Then w non-ε K since $V_p \cdot {}^2K = 0$. As w non-ε K there exists an arc $A_3 \subset M - w$ with end points z and p. It is easy to see that $\overline{V}_p \supset A_3$. From the set $A_1 + A_2 + A_3$ we may obtain a simple closed curve $J \subset \overline{V}_p$. Let C be the maximal cyclic set containing J. In the preceding paragraph we saw that $C \cdot {}^2K \neq 0$. But $C \subset \overline{V}_p \subset U_p$ and $U_p \cdot {}^2K = 0$. Then the assumption that p non-ε ${}^2\overline{K}$ leads to an absurdity. Hence $E \subset {}^2\overline{K}$. But $M = \overline{E}$. Then $M = {}^2\overline{K}$, which is the desired relation.

The condition is sufficient. From the result of §4 we have the set $F = M - K \subset \overline{E}$ and thus $\overline{F} \subset \overline{E}$. If C is a maximal cyclic set of M, all but a countable number of points of C belong to $M - K$. Then $C \subset \overline{F}$. Let $p \varepsilon M - \overline{F}$ which is of order 2 in M and let U_p be a neighborhood of p. There exists a number $\epsilon_1 > 0$ such that $S(p, \epsilon_1) \cdot \overline{F} = 0$ and $S(p, \epsilon_1) \subset U_p$. If J is any simple closed curve of M, $S(p, \epsilon_1) \cdot J = 0$. As $p \varepsilon K$ there is an arc $A \subset U_p$ and containing p as an interior point. Since ${}^2\overline{K} = M$, there is a point $q_1 \varepsilon {}^2K \cdot S(p, \epsilon_1)$. If $q_1 \varepsilon A$, let $x_1 = q_1$. In this case there is a component H_1 of $M - A$ such that $\overline{H}_1 \cdot A = q_1 = x_1$. If q_1 non-ε A, let H_1 be the component of $M - A$ containing q_1. The set $\overline{H}_1 \cdot A$ is a single point x_1 for otherwise there exists a simple closed curve J such that $J \cdot S(p, \epsilon_1) \neq 0$. Since p is of order 2, $p \neq x_1$. There exists a number $0 < \epsilon_2 < \frac{1}{2}\epsilon_1$ such that $S(p, \epsilon_2) \cdot \overline{H}_1 = 0$. Let $q_2 \varepsilon {}^2K \cdot S(p, \epsilon_2)$. If $q_2 \varepsilon A$, let $q_2 = x_2$ and there is a component H_2 of $M - A$ such that $\overline{H}_2 \cdot A = q_2 = x_2$. If q_2 non-ε A, let H_2 be the component of $M - A$ containing q_2 and let $x_2 \varepsilon \overline{H}_2 \cdot A$. As above $x_2 = \overline{H}_2 \cdot A$ since $J \cdot S(p, \epsilon_1) = 0$ for every simple closed curve J. Continue this process indefinitely. In general there exists a number $0 < \epsilon_n < \frac{1}{2}\epsilon_{n-1}$ such that $S(p, \epsilon_n) \cdot (\overline{H}_1 + \overline{H}_2 + \cdots + \overline{H}_{n-1}) = 0$. Let $q_n \varepsilon {}^2K \cdot S(p, \epsilon_n)$ and define x_n and H_n as above. Then $x_n \neq p$.

From the connectedness im kleinen of M we have

$$\lim_{n \to \infty} x_n = p, \qquad \lim_{n \to \infty} d(H_n) = 0. \tag{4}$$

From this it follows that there exists an integer m such that $H_m \subset S(p, \epsilon_1)$. The locally connected continuum $\overline{H}_m$ is acyclic

since $J \cdot S(p, \epsilon_1) = 0$ for each simple closed curve of M. Then every node of $\overline{H}_m$ is an end point of $\overline{H}_m$. This set contains at least two nodes and thus there is a point $z \neq x_m$ which is an end point of $\overline{H}_m$. It is clear that $z \varepsilon E$. Then $z \subset \overline{H}_m \subset S(p, \epsilon_1) \subset U_p$. Therefore $p \epsilon \overline{E}$.

We have $M - K + K^2 \subset \overline{E}$, and $M - (M - K + K^2) = {}^2K$. By a theorem of G. T. Whyburn,* the set ${}^2K$ is countable and thus contains no open subset of M. For this reason ${}^2K \subset \overline{E}$.

7. *The Necessity of Local Connectivity.* We shall show now that the condition of the preceding theorem is neither necessary nor sufficient unless M is locally connected. That the condition is not sufficient follows from the first two properties of a continuum M described in the next paragraph.

There exists a continuum M with the following properties: (1) *M has no end point,* (2) *the cut points of order >2 are dense in M,* (3) *the set of cut points of order 2 is an uncountable zero-dimensional set which is dense in M.*

Let a, b, c be the points $(0, 0)$, $(1, 1)$, $(1, -1)$. Let $M_1 = abc$, where by such a symbol we will mean the set of all points of the triangle plus all interior points. Let x_i be the number $1 - 1/2^i$. Let y_i be the ith number in the series $0, \frac{1}{2}, 0, -7/8, \cdots$, that is, $y_i = 0$, x_{i-1} or $-x_{i-1}$ according as i is of the form $2n+1$, $4n+2$ or $4n$. Let p_i $(i = 1, 2, \cdots)$ be the point (x_i, y_i). Let u_i and v_i $(i = 0, 1, 2, \cdots)$ be the upper and lower end points of a vertical segment of length $1/2^i$ with p_{i+1} as mid-point. Let $p_0 = a$. Let

$$M_2 = bc + \sum_{i=0}^{\infty} p_i u_i v_i.$$

Now let T_i be the collineation such that $T_i(a) = p_i$, $T_i(b) = u_i$, $T_i(c) = v_i$. Let

$$M_3 = \sum_{i=0}^{\infty} T_i(M_2).$$

In general let Δ be a maximal triangle of M_{n-1} and p_Δ, u_Δ and v_Δ be the left-, upper right-, and lower right-hand vertices of Δ. Let T_Δ be the collineation such that $T_\Delta(a) = p_\Delta$, $T_\Delta(b) = u_\Delta$, $T_\Delta(c) = v_\Delta$, and let

* G. T. Whyburn, *Concerning collections of cuttings of connected point sets.* this Bulletin, vol. 35 (1929), pp. 87–104, Theorem 13.

$$M_n = \sum T_\Delta(M_2)$$

for every $\Delta \subset M_{n-1}$. Now let

$$G = \prod_{n=1}^{\infty} \overline{M}_n.$$

The set G is a continuum such that every vertical line $x=k(0 \leqq k \leqq 1)$ intersects it in a single point or in an interval which is a side of one of the triangles Δ at some stage. The mid-point of each of these intervals (except bc) is a cut point of G of infinite order due to the oscillation of the set to the left of the point. It is easily seen that these cut points are dense in G. The point a is the only end point of G. Let H be the reflection of G in the y-axis, and let $M=G+H$. Then the continuum M has properties (1) and (2).

Except for a countable number of values of k, the line $x=k(-1 \leqq k \leqq 1)$ intersects M in a single point. Let q be any such point. The point $q=a$ is obviously a cut point of order 2 as it is of order 1 in both G and H. If $q \neq a$, then q is interior to some such point. Then $q=a$ is obviously a cut point of order 2 as it is of order 1 in both G and H. If $q \neq a$, then q is interior to some triangle Δ of M_n at each stage. If $\epsilon > 0$, there exists an integer n such that the triangle Δ of M_n containing q is of diameter $< \frac{1}{2}\epsilon$ and a non-ε Δ. The triangle Δ contains just two limit points of that part of M outside Δ (its left-hand vertex and the mid-point of the vertical side). From this it is easy to see that we may obtain a quadrilateral Q of diameter $< \epsilon$ which has just these two points in common with Δ, encloses the rest of Δ, and has in its exterior $M-\Delta$. Then the interior of Q is a neighborhood of q of diameter $< \epsilon$ whose boundary has just two points in common with M. Hence q is of order 2. That dim $K^2=0$ follows at once for both points of $M \cdot B(Q)$ belong to ${}^2K$. This gives $\dim_q K^2=0$ for $q \neq a$. It is also easy to see that $\dim_a K^2=0$.

We shall proceed to show now that the condition of the preceding theorem is not necessary unless M is locally connected, that is,

There exists a continuum M, which contains no cut point of order >2, and yet the end points are dense in M.

Let $[r_n]$ denote the rational points in the interval (0, 1),

except 0, arranged in a sequence. Let $f_n(x) = (1/n^2)[\sin 1/(x-r_n)]$ for $x \neq r_n$; $f_n(x) = 0$ for $x = r_n$ $(0 \leqq x \leqq 1)$. Let

$$F(x) = \sum_{n=1}^{\infty} f_n(x).$$

Let H be the set composed of the graph of $F(x)$ and its limit points.* The following properties of H may be easily verified: (a) H is a continuum, (b) every line $x = k (0 \leqq k \leqq 1)$ intersects H in either a point or an interval, (c) every such point is a cut point of H of order 2, (d) the set of all such intervals is a countable set, (e) H contains no cut point of order >2, (f) the point $(0, F(0))$ is the single end point of H.

Now we may transform the set H into a set M_1 such that (1) the point $(0, F(0))$ of H becomes the point $a = (0, 0)$, (2) the upper and lower end points of the interval of H containing $(1, F(1))$ become the points $b = (1, 1)$ and $c = (1, -1)$, (3) every point of M_1 lies in or on the triangle abc, (4) the properties (a)–(f) remain true for M_1. In fact we may make this transformation so as not to change the x-coordinate of any point.

For each interval I_n of M_1 let W_n be a wedge-shaped region bounded by two planes through I_n making equal angles with the xy-plane, two planes through the end points of I_n perpendicular to I_n and a plane parallel to the xy-plane. Evidently the regions W_n may be taken so that

$$\overline{W}_i \cdot \overline{W}_j = 0 \ (i \neq j), \text{ and } d(W_n) < 1/n.$$

Let $(a_n, b_n, 0)$ and $(a_n, c_n, 0)$ be the end points of $I_n (b_n > c_n)$, and let d_n be a number such that the point $(a_n, \frac{1}{2}(b_n + c_n), d_n)$ lies in the interior of W_n. Let Ψ_n be the transformation $x' = a_n$, $y' = \frac{1}{2}(b_n + c_n) + \frac{1}{2}(b_n - c_n)y$, $z' = -d_n x + d_n$. Let

$$M_2 = \sum_{n=1}^{\infty} \Psi_n(M_1).$$

* See W. A. Wilson, *On the structure of a continuum, limited and irreducible between two points*, American Journal of Mathematics, vol. 48 (1926), p. 162. The continuum H may also be constructed by methods similar to the previous example by building up oscillations both right and left. However the description becomes rather intricate. Also the previous example may be set up by the Cantor method of condensation of singularities but its properties appear somewhat clearer by the use of the triangles Δ.

Let I_{nm} be the intervals of $\Psi_n(M_1)$ except for I_n, and for each I_{nm} we take a wedge-shaped region W_{nm} such that $\overline{W}_{ni} \cdot \overline{W}_{nj} = 0 (i \neq j)$, $d(W_{nm}) < 1/(n+m)$, $W_{nm} \cdot (M_1 + M_2) = I_{nm}$, $\overline{W}_{nm} \subset$ interior W_n. Now we define a linear transformation Ψ_{nm} such that $\Psi_{nm}(M_1)$ is a set homeomorphic with M_1 and lying in W_{nm} except for I_{nm}, just as above. Let

$$M_3 = \sum_n \sum_m \Psi_{nm}(M_1).$$

Continue this process indefinitely. Now let

$$M = \sum_{i=1}^{\infty} M_i.$$

The set $\overline{M}$ is the required example. To simplify the description as far as possible we have defined $\overline{M}$ as lying in euclidean three-space. However, on careful examination of $\overline{M}$ it may be seen that $\overline{M}$ can be mapped in the plane.

8. THEOREM. *If the end points are dense in M, then* dim $M^2 \leqq 0$.

In a paper not yet published,* I have proved that for the set M^2, that is, the points of M of order 2, we have the following relation:

$$M^2 = H + G,$$

where (1) dim $H \leqq 0$, (2) G is a vacuous or countable set of arcs A_i, (3) each arc-segment A_i is an open subset of M. Now if we suppose that dim $M^2 > 0$, it follows from this result that the set G contains at least one arc A_1. Let p be an interior point of A_1. From property (3) there exists a neighborhood $U_p \subset A_1 \subset M^2$. Hence $U_p \cdot E = 0$ contrary to the hypothesis of our theorem.

COROLLARY. *If the end points are dense in M, then* dim $K^2 \leqq 0$.

9. THEOREM. *In order that the end points of a locally connected continuum M be dense in M it is necessary and sufficient that the cut points be dense in M and* dim $K^2 \leqq 0$. $[(\overline{E} = M) \equiv (\overline{K} = M) \cdot (\dim K^2 \leqq 0).]$

* *On the regular points of a continuum.* This paper has been submitted to the Transactions of this Society. See an abstract in this Bulletin, vol. 36 (1930), p. 485.

The conditions of the theorem are necessary by the results of §3 and the corollary of §8. We may show that the conditions are sufficient by an argument following closely that used in proving the sufficiency of the condition in §6.

The first example of §7 shows that the conditions of the theorem are not sufficient if we do not specify that M is locally connected.

THE UNIVERSITY OF MICHIGAN

ON THE GENERALIZATION OF TRIGONOMETRIC IDENTITIES IN ARITHMETICAL PARAPHRASING*

BY H. T. ENGSTROM†

1. *Introduction.* Identities of the type

$$\sum_{s=1}^{m}\alpha_s \sin a_s x \equiv \sum_{t=1}^{n}\beta_t \sin b_t x, \tag{1}$$

where α_s, a_s, β_t, b_t are rational integers, arise in the comparison of like powers of the modulus when an elliptic function is represented in more than one way by trigonometric series. The following theorem is used in obtaining arithmetical results from such identities.

THEOREM 1. *If $g(x)$ is an arbitrary, single-valued, odd function, defined for $x=a_s$, $s=1, 2, \cdots, m$, and $x=b_t$, $t=1, 2, \cdots, n$, then* (1) *implies*

$$\sum_{s=1}^{m}\alpha_s g(a_s) = \sum_{t=1}^{n}\beta_t g(b_t). \tag{2}$$

Similarly, for cosines, we have the following statement.

THEOREM 2. *If $f(x)$ is an arbitrary, single-valued, even function, defined for $x=a_s$, $s=1, 2, \cdots, m$, and $x=b_t$, $t=1, 2, \cdots, n$, then*

$$\sum_{s=1}^{m}\alpha_s \cos a_s x \equiv \sum_{t=1}^{n}\beta_t \cos b_t x \tag{3}$$

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* Presented to the Society, April 5, 1930.

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implies

$$\sum_{s=1}^{m}\alpha_s f(a_s) = \sum_{t=1}^{n}\beta_t f(b_t). \tag{4}$$

The method of proof for these theorems suggested by Liouville* assumes that the functions may be represented by a Fourier series. Nazimoff† also makes use of the Fourier series property. The method is perfectly general since the functions $f(x)$ and $g(x)$ are defined for only a finite number of values of the argument and hence by interpolation we may represent them by a Fourier series. It is desirable, however, to give a purely algebraic proof of the replacement principle. E. T. Bell,‡ in his fundamental paper on arithmetical paraphrases, avoids the Fourier series by making use of the lemmas which follow.

LEMMA 1. *If the a_s are rational integers $\gtreqless 0$ and the b_t rational integers $\gtrless 0$, and if for all integral values >0 of k*

$$\sum_{s=1}^{m}a_s^{2k} = \sum_{t=1}^{n}b_t^{2k},$$

then (i) *$m \geqq n$ and precisely $(m-n)$ of the $a_s = 0$;* (ii) *if, without loss of generality, the n non-zero a_s are $a_1, a_2, \cdots, a_n$, then the $a_1^2, a_2^2, \cdots, a_n^2$ are a permutation of the $b_1^2, b_2^2, \cdots, b_n^2$.*

LEMMA 2. *If the a_s, b_t are integers >0, and if there is an infinity of odd integers $k>0$, for which*

$$\sum_{s=1}^{m}a_s^k = \sum_{t=1}^{n}b_t^k,$$

then $m=n$, and the a_s are a permutation of the b_t.

Bell (loc. cit.) has given a simple algebraic proof of Lemma 1. Lemma 2, however, has not been proved by purely algebraic means. The proofs given by E. Swift§ and C. F. Gummer‖ depend on continuity considerations. In the following paper

* *Note de M. Liouville*, Journal de Mathématiques, vol. 7 (1862), p. 48.

† Nazimoff, Annales Scientifiques de l'École Normale, vol. 3 (1868), p. 149.

‡ Bell, Transactions of this Society, vol. 22 (1921), p. 17.

§ E. Swift, American Mathematical Monthly, vol. 24 (1917), p. 288.

‖ C. F. Gummer, Transactions of this Society, vol. 23 (1922), p. 280.

the author presents a direct algebraic proof of Theorems 1 and 2 based on the irreducibility of the cyclotomic equation.

2. *Cyclotomic Fields.* Consider the cyclotomic field K of degree $p-1$ defined by a root of the equation

$$\phi(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 = 0, \tag{5}$$

where p is an odd prime. From the irreducibility of $\phi(x)$ the following theorem is obvious.

THEOREM 3. *The algebraic integers*

$$e^{2k\pi i/p},\ k = \pm 1,\ \pm 2, \cdots, \pm \frac{p-1}{2}, \tag{6}$$

are linearly independent for rational coefficients.

We shall now consider Theorem 1. Let us choose an odd prime p such that

$$\frac{p-1}{2} > \max(m, n, |a_s|, |b_t|). \tag{7}$$

Substituting $x = 2\pi/p$ in (1) and replacing the sines by exponentials we obtain

$$\sum_{s,t}\left\{\alpha_s e^{2\pi a_s i/p} - \alpha_s e^{-2\pi a_s i/p} - \beta_t e^{2\pi b_t i/p} + \beta_t e^{-2\pi b_t i/p}\right\} = 0. \tag{8}$$

From (7), the terms in (8) are either zero or belong to the set (6). Hence the coefficient of $e^{2\pi ki/p}$, $k = \pm 1, \pm 2, \cdots, \pm (p-1)/2$ in (8) must vanish. Let us consider the three cases: (i) the a_s and b_t all different from zero and of like sign; (ii) some a_s or b_t equal to zero and the remainder of like sign; (iii) the general case.

In case (i) it is seen that the coefficient of $\sin kx$, $k = \pm 1$, $\pm 2, \cdots, \pm (p-1)/2$, in (1) is precisely that of $e^{2\pi ki/p}$ in (8) and hence must vanish. The following generalization of Theorem 1 follows immediately.

THEOREM 4. *If a_s and b_t satisfy* (i) *and $F(x)$ is an arbitrary function defined for $x = a_s$ and $x = b_t$, then* (1) *implies*

$$\sum_{s=1}^{m} \alpha_s F(a_s) = \sum_{t=1}^{n} \beta_t F(b_t).$$

We now turn to case (ii). Let $F_0(x)$ be any function for which $F_0(0)=0$. We substitute $F_0(a_s)$ and $F_0(b_t)$ respectively in place of $\sin(a_s x)$ and $\sin(b_t x)$ in (1). If $k \not\equiv 0$, the coefficient of $F_0(k)$ will be precisely that of $e^{2\pi ki/p}$ in (8) and hence vanishes. If $k=0$, by definition $F_0(k)=0$, and hence we have the following theorem.

THEOREM 5. *If a_s, b_t satisfy* (ii) *and $F_0(x)$ is an arbitrary function defined for $x=a_s$ and $x=b_t$ and vanishing for $x=0$, then* (1) *implies*

$$\sum_{s=1}^{m} \alpha_s F_0(a_s) = \sum_{t=1}^{n} \beta_t F_0(b_t).$$

Let us consider the general case (iii) and let $g(x)$ be an arbitrary odd function. If a_s is positive we replace $\sin(a_s x)$ in (1) by $g(a_s)$, if negative by $-g(-a_s)$. Similarly for b_t. The resulting coefficient of $g(k)$, $k=1, 2, \cdots, (p-1)/2$, is seen to be precisely that of $e^{2\pi ki/p}$ in (8) and hence is zero. Since $x=0$ implies $g(x)=0$ and $g(x)=-g(-x)$, Theorem 1 follows.

Let us consider Theorem 2. We again choose p in accordance with (7). Substituting $x=2\pi/p$ in (3) and expressing in terms of exponentials we obtain

$$(9) \quad \sum_{s,t} \left\{ \alpha_s e^{2\pi a_s i/p} + \alpha_s e^{-2\pi a_s i/p} - \beta_t e^{2\pi b_t i/p} - \beta_t e^{-2\pi b_t i/p} \right\} = 0.$$

The reasoning used for Theorem 1 must be modified slightly on account of the rational terms which may enter for vanishing a_s or b_t. Since (6) are the roots of (5) we have

$$\sum_{k=\pm 1, \pm 2, \cdots, \pm (p-1)/2} e^{2\pi ki/p} = -1.$$

Furthermore, from Theorem 3, this representation of -1 as a rational linear combination of (6) is unique. Hence we have the following theorem.

THEOREM 6. *If c_k and R are rational, then*

$$\sum_{k=\pm 1, \pm 2, \cdots, \pm (p-1)/2} c_k e^{2\pi ki/p} = R$$

implies $c_k = -R$, $k = \pm 1, \pm 2, \cdots, \pm (p-1)/2$.

It follows from this theorem and (7) that the coefficients of $e^{2\pi ki/p}$ in (9) and also the rational term must vanish. Let us now consider two cases: (i) a_s and b_t all of like sign or zero; and

(ii) the general case. In case (i) let $F(x)$ be an arbitrary function defined for $x=a_s$ and $x=b_t$. We substitute $F(a_s)$ and $F(b_t)$ for $\cos(a_s x)$ and $\cos(b_t x)$ respectively. The coefficient of $F(l)$, $l=0, 1, 2, \cdots, (p-1)/2$, will be precisely that of $e^{2\pi li/p}$ in (9) and hence vanishes. Hence we have the following theorem:

THEOREM 7. *If the a_s, b_t in* (3) *satisfy* (i) *and $F(x)$ is an arbitrary function defined for $x=a_s$ and $x=b_t$, then* (3) *implies*

$$\sum_{s=1}^{m} \alpha_s F(a_s) = \sum_{t=1}^{n} \beta_t F(b_t).$$

For the general case (ii), let $f(x)$ be an arbitrary even function. If a_s is positive, substitute $f(a_s)$ in place of $\cos(a_s x)$ in (3), if negative, substitute $f(-a_s)$. Similarly for b_t. Then, as above, the coefficient of $f(l)$, $l=0, 1, 2, \cdots, (p-1)/2$, will be precisely that of $e^{2\pi li/p}$ in (9) and hence vanishes. Theorem 2 follows immediately.

3. *Removal of Conditions.* The condition that α_s, a_s, β_t, b_t be rational integers may be easily lessened. If a_s and b_t are of the form $A_s N$ and $B_t N$ respectively, where A_s and B_t are rational and N is any real number, the transformation $x=MNy$, where M is a common multiple of the denominators of A_s and B_t, reduces the problem to the cases already considered. The theorems may also be generalized to algebraic α_s and β_t. We may suppose them algebraic integers since denominators may be eliminated by cross-multiplication and we may suppose that they all belong to an algebraic field K_1 of degree r. We shall show that p may be chosen so that (5) is irreducible in K_1.

By writing

$$\phi(x) = \prod_{k=\pm 1, \pm 2, \cdots, \pm (p-1)/2} (x - e^{2\pi ki/p})$$

it is seen that $\phi(x)$ may be reducible only in those fields which are subfields of K. We first place on p the condition

$$\left(\frac{p-1}{2}, r\right) = 1 \tag{10}$$

which insures that r and $p-1$ have at most the common divisor 2, that is, K and K_1 have at most a quadratic subfield in common. But the field K_1 contains only a finite number of distinct

quadratic subfields. Suppose that these fields are generated by $c_1^{1/2}, c_2^{1/2}, \cdots, c_\mu^{1/2}$. Since the only quadratic field in which $\phi(x)$ is reducible is that generated by

$$[(-1)^{(p-1)/2} \cdot p]^{1/2},$$

if we impose on p the further condition

$$(p, c_1, c_2, \cdots, c_\mu) = 1, \tag{11}$$

it follows that K and K_1 will have no common subfield and hence $\phi(x)$ will be irreducible in K_1. Hence a linear combination of (6) with coefficients in K_1 is zero if, and only if, each coefficient is zero. Theorems 1, 4, and 5 generalize directly. Theorem 6 takes the following form.

THEOREM 8. *If p satisfies* (10) *and* (11), *the relation*

$$\sum_k \theta_k e^{2\pi ki/p} = \rho,$$

where θ_k and ρ belong to K_1, implies $\theta_k = -\rho$, $k = \pm 1, \pm 2, \cdots, \pm(p-1)/2$.

Theorems 1 and 7 generalize immediately.

To extend the theorems to arbitrary complex α_s, β_t let us suppose that some of them are not algebraic. We may then replace the α_s, β_t by linear combinations of a set $1, \pi_1, \pi_2, \cdots, \pi_\nu$ with algebraic coefficients, where $1, \pi_1, \pi_2, \cdots, \pi_\nu$ are linearly independent for algebraic coefficients. Hence, for similar choice of p, (8) and (9) imply $\nu+1$ equations of the same type with algebraic coefficients and the theorems follow. The results are combined in the following theorem.

THEOREM 9. *The theorems of this paper are valid for arbitrary complex α_s, β_t and for a_s, b_t of the form $A_s N, B_t N$ respectively, where A_s, B_t are rational and N is an arbitrary complex number.*

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AN ELEMENTARY THEOREM ON MATRICES*

BY M. H. INGRAHAM

This note applies some elementary algebraic theory to the theory of matrices, securing a generalization of the theorem:

If $y_1, \cdots, y_n$ are elements of a field F corresponding to n distinct elements $x_1, \cdots, x_n$ in F, there exists one and only one polynomial f of degree less than n such that $f(x_i) = y_i (i = 1, \cdots, n)$.

As is well known, every finite square matrix m with elements in a field F satisfies its characteristic equation $|m - \lambda I| = 0$, and hence satisfies a unique equation, $g(\lambda) = 0$, of minimum degree with leading coefficient unity. Moreover, if for two polynomials f and h, $f(m) = h(m)$, then $f(\lambda) \equiv h(\lambda) \bmod g(\lambda)$ and conversely.

We seek an answer to the question "Given finite square matrices $m_1, \cdots, m_n$ and polynomials $h_1, \cdots, h_m$, under what conditions can a polynomial f be found such that $f(m_i) = h_i(m_i)$ $(i = 1, \cdots, n)$?"

If the minimum equations of the m_i are $g_i(\lambda) = 0, (i = 1, \cdots, n)$. the above question is equivalent to asking, under what conditions the congruences

$$f(\lambda) \equiv h_i(\lambda) \mod g_i(\lambda), \qquad (i = 1, \cdots, n),$$

have a solution.

Standard works on number theory discuss this question and give its solution in an elementary fashion. In particular, there are always solutions when $g_1, \cdots, g_n$ are relatively prime.

From considerations of this known work on congruences, many theorems, two of which follow, may be translated at once to their matrix form.

THEOREM 1. *If there exists a polynomial f in a field F such that for n finite square matrices $m_1, \cdots, m_n$ with elements in F, and n polynomials $h_1, \cdots, h_n$ in F, $f(m_i) = h_i(m_i)$, $(i = 1, \cdots, n)$, then there exists one and only one such f of degree lower than that of k, the least common multiple of the minimum equations of $m_1, \cdots,$*

* Presented to the Society, September 11, 1930.

m_n, all other solutions being obtained from this solution by the addition of multiples of k.

THEOREM 2. *If the minimum equations of n finite square matrices m_1 to m_n with elements in a field F are relatively prime then for any set of n polynomials $h_1, \cdots, h_n$, in F, a polynomial f may be found such that*

$$f(m_i) = h_i(m_i), \qquad (i = 1, \cdots, n).$$

These theorems specialize to the above mentioned algebraic theorem since each x_i is a one by one matrix with minimum equation $\lambda - x_i = 0$.

It should be noted that in the above discussion no restriction as to the field in which the elements of the matrices may lie is made, nor are the m_i necessarily of the same order.

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PROBLEMS OF THE CALCULUS OF VARIATIONS WITH PRESCRIBED TRANSVERSALITY CONDITIONS*

BY LINCOLN LA PAZ†

1. *Introduction.* Problems of the calculus of variations in the plane for which a prescribed relation exists between the directions of the extremals and the transversals were first studied by Stromquist‡ and Bliss.§ Recently Rawles,‖ using a method based on properties of the Hilbert invariant integral, has given an interesting treatment of the analogous problem in $(x, y_1, \cdots, y_n)$-space.

In the present paper the latter problem is attacked from a quite different point of view.¶ The method here used avoids a restrictive hypothesis made by Rawles with regard to the ex-

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* Presented to the Society, August 29, 1929.

† National Research Fellow, 1928–1929.

‡ Stromquist, Transactions of this Society, vol. 7 (1906), p. 181; Annals of Mathematics, (2), vol. 9 (1907), p. 57.

§ Bliss, Annals of Mathematics, (2), vol. 9 (1907), p. 134.

‖ Rawles, Transactions of this Society, vol. 30 (1928), pp. 765–784.

¶ The possibility of approaching the problem from this viewpoint was suggested to the writer by G. A. Bliss.

istence of an n-parameter family of Mayer fields;* and the results of his paper are supplemented by a proof that his formula† $f=ge^H$ actually gives the most general non-singular integrand function f compatible with a prescribed transversality relation.

2. *Properties of the Transversality Coefficients of a Non-Singular Problem of the Calculus of Variations.* For a problem of minimizing the integral

$$(1) \qquad I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx,$$

an extremal E through the point $(x, y_1, \cdots, y_n)$ in the direction $(1:y_1': \cdots :y_n')$ is said to be cut transversally by the hyperplane of directions $(dx:dy_1: \cdots :dy_n)$ through this point in case

$$(2) \qquad (f - y_\mu' \partial f/\partial y_\mu')dx + (\partial f/\partial y_\mu')dy_\mu = 0.$$

In (2) the arguments in f and its derivatives are $x, y_1, \cdots, y_n$, $y_1', \cdots, y_n'$ and, as elsewhere in this paper, a repeated Greek letter is an umbral index indicating a summation with the range 1 to n unless otherwise specified. This definition of transversality is the usual one: it makes no appeal to the notion of a field or the transversal hypersurfaces thereof and defines what is meant by transversality for every set $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ in the fundamental region R of the integral (1)‡.

If (1) is a non-singular problem, that is to say, if there exists a region R in which $D = |\partial^2 f/\partial y_i' \partial y_j'|$, $(i, j = 1, \cdots, n)$, is everywhere different from zero,§ then the quantity $(f - y_\mu' \partial f/\partial y_\mu')$

* Rawles, loc. cit., p. 774.

† Loc. cit., p. 778.

‡ Consult in this connection Bliss, *Calculus of Variations*, 1925, p. 129. In the present paper we assume that f is of class C''' in a region R of $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-values. For a definition of the term class as here used see Bolza, *Vorlesungen über Variationsrechnung*, p. 13.

§ Problems for which $D \neq 0$ have been called *regular* by Goursat (*Cours d'Analyse Mathématique*, vol. III, 1923, p. 561) and *ordinary* by Hadamard (*Leçons sur le Calcul des Variations*, vol. I, 1910, p. 68). The term *non-singular* has been employed in lectures by G. A. Bliss, who characterizes a problem as regular only in case the quadratic form with matrix $\|\partial^2 f/\partial y_i' \partial y_j'\|$ is positive definite. This latter usage seems preferable since it requires that in higher space as in the plane the Legendre condition in stronger form shall hold for regular problems.

is not identically zero. For if f satisfies the equation $f-y'_\mu \partial f/\partial y'_\mu = 0$ it can be shown by consideration of the associated symmetric system

$$dy'_1/y'_1 = \cdots = dy'_n/y'_n = df/f \tag{3}$$

that the most general non-singular integral f of $f-y'_\mu \partial f/\partial y'_\mu = 0$ is given by

$$f = y'_1 F(u_2, \cdots, u_n, x, y_1, \cdots, y_n), \qquad u_k = y'_k/y'_1, \quad k = 2, \cdots, n, \tag{4}$$

where F is an arbitrary function of its arguments. But for the integrand function f in (4) it is found that

$$D = \begin{vmatrix} \dfrac{\partial^2 F}{\partial u_\beta \partial u_\gamma} \dfrac{y'_\beta y'_\gamma}{(y'_1)^3}, & -\dfrac{\partial^2 F}{\partial u_\beta \partial u_\epsilon} \dfrac{y'_\beta}{(y'_1)^2} \\ -\dfrac{\partial^2 F}{\partial u_\tau \partial u_\beta} \dfrac{y'_\beta}{(y'_1)^2}, & \dfrac{\partial^2 F}{\partial u_\tau \partial u_\epsilon} \dfrac{1}{y'_1} \end{vmatrix}, \tag{5}$$

where $\tau, \epsilon, \beta, \gamma = 2, \cdots, n$ and β and γ are umbral. The determinant on the right in (5) is easily shown to be identically zero. Hence a problem (1) with an integrand function (4) is always singular.

Assume (1) to be a non-singular problem and restrict attention to that portion of the fundamental region R in which the quantities f and $(f-y'_\mu \partial f/\partial y'_\mu)$ are different from zero. In this subregion of R the ratios

$$t_i = (\partial f/\partial y'_i)/(f - y'_\mu \partial f/\partial y'_\mu), \qquad (i = 1, \cdots, n), \tag{6}$$

are well defined and of class C''. These ratios will be called the *transversality coefficients* of the problem (1).

It is found that

$$1 + y'_\mu t_\mu = f/(f - y'_\mu \partial f/\partial y'_\mu). \tag{7}$$

Hence in the subregion of R specified, the transversality coefficients t_i necessarily satisfy the inequality

$$1 + y'_\mu t_\mu \neq 0. \tag{8}$$

Geometrically this means that the extremal through the point $(x, y_1, \cdots, y_n)$ in the direction $1 : y'_1 : \cdots : y'_n$ is not tangent to the hyperplane with normal $1 : t_1 : \cdots : t_n$ in this point.

Finally it is easy to verify that

$$(9) \qquad \left| \partial t_i/\partial y_j' \right| = \frac{\left| \partial^2 f/\partial y_i' \partial y_j' + t_i y_\mu' \partial^2 f/\partial y_\mu' \partial y_j' \right|}{(f - y_\mu' \partial f/\partial y_\mu')^n}, \qquad (i, j = 1, \cdots, n).$$

The value of the determinant on the right in (9) is given by the product of the determinants D and $\left|\delta_j^i + y_i' t_j\right|$, the latter of which is found to be equal to $1+y_\mu' t_\mu$. Hence

$$(10) \qquad \left| \partial t_i/\partial y_j' \right| = fD/(f - y_\mu' \partial f/\partial y_\mu')^{n+1},$$

and therefore in the subregion of R under consideration the Jacobian of the t_i with respect to the y_j' is different from zero.

3. *Conditions for a Set of n Functions to be Transversality Coefficients.* The condition (2) establishes a certain relation between the lineal element $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ through the point $(x, y_1, \cdots, y_n)$ and the hyperplane of directions $(dx:dy_1: \cdots :dy_n)$ with normal $(1:t_1: \cdots :t_n)$ through this point. If n given functions $T_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$, $i=1, \cdots, n$, of class C'' and satisfying the inequalities $1+y_\mu' T_\mu \neq 0$, $\left|\partial T_i/\partial y_j'\right| \neq 0$ in a region S of $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-values are to be the transversality coefficients of a non-singular problem (1) in S then the equations

$$(11) \qquad dx + T_\mu dy_\mu = 0,$$

and (2) must define the same hyperplane of directions $(dx:dy_1: \cdots :dy_n)$ for all sets $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ in S. Hence corresponding coefficients in these two relations must be proportional. Consequently a set of n functions T_i as described can be transversality coefficients of a non-singular problem (1) only in case there exists a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ satisfying the system of equations

$$(12) \qquad f - y_\mu' \partial f/\partial y_\mu' = h, \quad \partial f/\partial y_i' = hT_i, \qquad (i = 1, \cdots, n),$$

and different from zero in S. In (12) $h \neq 0$ is a function of $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$. Elimination of h in (12) leads to the following system of non-homogeneous partial differential equations which must be satisfied by f regarded as a function of the n independent variables $(y_1', \cdots, y_n')$ and the $n+1$ parameters $(x, y_1, \cdots, y_n)$:

$$T_k f - (\delta_k^\beta + T_k y_\beta')(\partial f/\partial y_\beta') = 0, \qquad (k = 1, \cdots, n). \tag{13}$$

To determine the integrability conditions for the system (13) we transform it into a system of equations linear and homogeneous in the first partial derivatives of a function $G(x, y_1, \cdots, y_n, y_1', \cdots, y_n', f)$ with $G_f \neq 0$ which defines f by means of the equation $G=$constant. The resulting system is

$$U_k G = (\delta_k^\beta + T_k y_\beta')(\partial G/\partial y_\beta') + T_k f(\partial G/\partial f) = 0. \tag{14}$$

The matrix of coefficients of (14) contains the nth order determinant $|\delta_k^i + y_i' T_k|$, $i, k = 1, \cdots, n$, which has the value $(1+y_\mu' T_\mu)$ and hence is different from zero in S. Since the n equations in (14) are therefore independent, it follows that if a single one of the commutators of this system is not a linear combination of the original equations the only solution for G is a constant and hence no solution f of (13) exists. Therefore the equations (13) are compatible if and only if the system (14) is a complete system.

A typical commutator of the system (14) is found to have the value

$$(U_m U_k)G = (\Gamma_{mk} y_\beta' + T_k \delta_m^\beta - T_m \delta_k^\beta)(\partial G/\partial y_\beta') + \Gamma_{mk} f(\partial G/\partial f), \tag{15}$$

where

$$\Gamma_{mk} = y_\nu'(T_m \partial T_k/\partial y_\nu' - T_k \partial T_m/\partial y_\nu') + (\partial T_k/\partial y_m' - \partial T_m/\partial y_k'). \tag{16}$$

In order that

$$(U_m U_k)G = r_\nu U_\nu G, \quad m, k = 1, \cdots, n, \quad m < k, \tag{17}$$

where the r_i $(i=1, \cdots, n)$ are coefficients of combination it is necessary that

$$\begin{aligned} \Gamma_{mk} y_i' + T_k \delta_m^i - T_m \delta_k^i &= r_\nu(\delta_\nu^i + T_\nu y_i'), \quad i = 1, \cdots, n, \\ \Gamma_{mk} f &= r_\nu T_\nu f. \end{aligned} \tag{18}$$

Since $f \neq 0$ it follows that

$$\begin{aligned} &r_i = 0 \text{ if } i \neq m, k, \\ &r_i = T_k \text{ if } i = m, \\ &r_i = -T_m \text{ if } i = k. \end{aligned} \tag{19}$$

Hence the $n(n-1)/2$ quantities Γ_{mk} are necessarily zero if (14) is a complete system. Conversely the conditions

$$\Gamma_{mk} = 0, \quad m, k = 1, \cdots, n, \quad m < k, \tag{20}$$

are sufficient conditions for the system (14) to be complete, for if all of the Γ_{mk} vanish then there actually exist coefficients r_i, namely those in (19), for which the relations (17) subsist. Taken in connection with the results of §2 this proves the following theorem.

THEOREM 1. *Necessary and sufficient conditions for n functions $T_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ to be the transversality coefficients of a non-singular problem* (1) *in a region S of $(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$-values are that in S the functions T_i be of class C'' and satisfy the inequalities $|\partial T_i/\partial y_j'| \neq 0$, $1+y_\mu' T_\mu \neq 0$ and the $n(n-1)/2$ relations* (20).

4. *Determination of the most general Integrand Function.* Suppose that n functions T_i which satisfy the conditions of Theorem 1 are given. What is the most general integrand function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ of a non-singular problem (1) which has the set T_i as its transversality coefficients? To answer this question consider the line integral

$$H = \int_0^{y'} T_\mu dy_\mu' / (1 + y_\mu' T_\mu) \tag{21}$$

in the $(y_1', \cdots, y_n')$-space. In (21) the symbols 0 and y' used as limits represent n-partite variables. It is found that necessary conditions for (21) to be independent of the path of integration are that the T_i satisfy the $n(n-1)/2$ relations*

$$R_{ik} = 0, \quad i, k = 1, \cdots, n, \quad i < k, \tag{22}$$

where

$$R_{mk} = y_\mu' \left[T_m \partial T_\mu/\partial y_k' - T_k \partial T_\mu/\partial y_m' + T_\mu(\partial T_k/\partial y_m' - \partial T_m/\partial y_k') \right] + (\partial T_k/\partial y_m' - \partial T_m/\partial y_k'). \tag{23}$$

These conditions are also sufficient conditions for independence of path in a space of suitably simple connectivity properties.

It can be shown that in terms of the Γ_{mk} introduced in (16)

$$R_{mk} = (1 + y_\mu' T_\mu)\Gamma_{mk} + y_\mu' T_k \Gamma_{\mu m} + y_\mu' T_m \Gamma_{k\mu}. \tag{24}$$

* Compare Rawles, loc. cit., p. 778.

Hence when all of the Γ_{mk} vanish, as they do for the set T_i under consideration, the conditions for the line integral H to be independent of the path in such a space are satisfied.

A particular solution of the homogeneous system (14) is

$$G = fe^{-H}. \tag{25}$$

Hence, since the general integral of a complete system of n equations in $n+1$ independent variables is an arbitrary function of a single particular integral of the system,* the most general solution of (14) is given by

$$G = F(fe^{-H}; x, y_1, \cdots, y_n), \tag{26}$$

where F is an arbitrary function of the arguments indicated. The most general non-singular solution f of the system of equations (13) is now obtained by solving for f the equation $F(fe^{-H}; x, y_1, \cdots, y_n) = 0$.† Hence we reach the following conclusion.

THEOREM 2. *If n functions $T_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ satisfy the conditions of Theorem* 1, *then there always exists a non-singular problem* (1) *with an integrand function f of class C''' which has the functions T_i as its transversality coefficients. The most general such problem has an integrand function*

$$f = g(x, y_1, \cdots, y_n)e^H \tag{27}$$

where g is different from zero and of class C''' but is otherwise an arbitrary function of its arguments, and where H has the value (21).

An integrand function f of the form (27) was found in an entirely different manner by Rawles. His argument although of much interest requires the decidedly restrictive hypothesis of the existence of an n-parameter family of Mayer fields; and, moreover, does not show that a function f of the form (27) is actually the most general integrand function of a problem (1) with a prescribed transversality condition.

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* Goursat, *Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre*, 1921, p. 71.

† Goursat, loc. cit., p. 95.

ON SURFACES IN SPACES OF FOUR AND FIVE DIMENSIONS

BY B. C. WONG

It is known that an algebraic surface in 4-space has four essential characteristics in terms of which all its other characteristics can be expressed. Severi* considers as essential the following: n, the order of the surface; a, the order of the tangent cone of its projection in a 3-space; t, the number of its apparent triple points; and n', the number of 3-spaces that can be constructed tangent to it and passing through a given point.

In order to know anything about a surface in 4-space, it is, then, necessary to know its four essential characteristics. Of course, n may always be assumed. Unless some independent means be found whereby the other three characteristics, a, t, n' can be calculated, we cannot know very much about the surface. It is our purpose in this paper to present a method for the independent determination of these three characteristics.

Let F'^n denote a surface of order n in 4-space, and if the surface is the projection of a surface of the same order in 5-space, let F^n denote the latter. We introduce three other characteristics of F'^n: b, the order of the cone of lines passing through a general point in S_4 and meeting F'^n twice; j, the number of tangent lines of F'^n passing through a given point; and d, the number of improper double points† on F'^n.

The seven characteristics are connected by the following relations:‡

$$a + 2b = n(n-1), \quad j + 2d = n(n-1) - a,$$
$$j = \tfrac{1}{4}[a(3n-4) - n(n-1)(n-2) + 6t - 2n'],$$
$$d = \tfrac{1}{8}[n(n-1)(n+2) - 3na - 6t + 2n'].$$

* Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a'suoi punti tripli apparenti*, Rendiconti di Palermo, vol. 15 (1901), pp. 33–51.

† An improper double point Q is one such that an S_3 through it meets the surface in a curve with an actual double point at Q and of the same deficiency as that of a general 3-space section of the surface.

‡ Severi, loc. cit., pp. 34–36.

From these relations we see that, n being assumed, if a, t, n' are known, j and d are determined. On the other hand, if we know t and any two of the three quantities a, j, d, we can calculate n'. Our method will, however, enable us to determine directly all the four quantities a, j, d, and t.

Consider a surface F^n of order n in S_5 and let it be the complete intersection of three hypersurfaces V_4^λ, V_4^μ, V_4^ν of orders λ, μ, ν respectively. We shall represent F^n symbolically* by means of $n=\lambda\mu\nu$ triads (x,y,z) whose elements x, y, z are to take on all the integral values from 1 to λ, μ, ν respectively. Thus, the surface F^8 common to three hyperquadric surfaces will be represented by the eight triads (1,1,1), (2,1,1), (1,2,1), (1,1,2), (1,2,2), (2,1,2), (2,2,1), (2,2,2). If one of the hypersurfaces, say V_4^ν, is a hyperplane S_4, the surface is a 4-space F'^n of order $n=\lambda\mu$ lying in S_4 and will be represented by the $\lambda\mu$ triads $(x,y,1)$.

Any triad taken alone in this representation of F^n represents a plane forming a part of F^n. Any pair of triads with two corresponding elements alike, as (1,1,1), (1,1,2) or (1,2,3), (4,2,3) represents a quadric surface or a pair of planes with a line in common. If two of the corresponding elements are different, as (1,1,1), (1,2,2) or (1,2,3), (4,1,3), we have a pair of planes intersecting in a point; but if all the three corresponding elements are different, the two planes have no point in common.

Now we determine the characteristic a. It is not difficult to see that a is the order of the hypersurface formed by the ∞^1 tangent 3-spaces of F^n that pass through a given plane α in S_5. Consider a quadric surface F^2 given in a general S_3 of S_5. Since S_3 has only a point in common with α, the ∞^1 tangent 3-spaces of F^2 passing through α form a hypersurface V_4^2 of order 2. If F^2 degenerates into two planes which must have a line in common, V_4^2 degenerates into the hyperplane determined by α and the line, counted twice. From this we infer that, in general, if F^n contains N double lines, the order of the hypersurface of tangent 3-spaces passing through a given plane is $a-2n$. Suppose F^n be composed entirely of planes, $n=\lambda\mu\nu$ in

* B. C. Wong, *On the number of apparent triple points of surfaces in space of four dimensions*, this Bulletin, vol. 35 (1929), pp. 339–343; and *On the number of apparent multiple points of varieties in hyperspace*, this Bulletin, vol. 36 (1930), pp. 102–106.

number. The number N of double lines on this degenerate F^n is equal to the number of pairs of triads in the representation of F^n with two corresponding elements alike. As F^n, degenerated in this way, cannot have proper tangent 3-spaces, we have $a-2N=0$ or $a=2N$.

To find N, we notice that, for all the integral values of x and y from 1 to λ and μ respectively, we have $\lambda\mu\nu(\nu-1)/2$ pairs of triads of the nature described above. Permuting λ, μ, ν and adding the results, we have $N=\lambda\mu\nu(\lambda+\mu+\nu-3)/2$ and, therefore $a=\lambda\mu\nu(\lambda+\mu+\nu-3)$* is the required formula for a.

Now we proceed to determine j. Consider a pair of triads with two corresponding different elements representing a pair of planes having a point in common. Any plane through this point and through a given line g in S_5 is to be considered a tangent plane of F^n through g, counted doubly. Any plane through g and tangent to a non-degenerate F^n gives rise to a line through a point in an S_4 and tangent to the projection F'^n in S_4. Then, the number of tangent planes of F^n passing through g which is equal to the number of tangent lines of F'^n passing through a given point in S_4 is the number j we are seeking. But if F^n be decomposed into n planes, there will be no proper tangent planes. Through a given line pass a certain number, N', of planes each passing through a point in which two planes of F^n as represented above intersect. As each such plane counts twice as a tangent plane of F^n, we have $j=2N'$.

To calculate N', we find that, for a fixed value of x, the number of pairs of triads such that the y- and z-elements of one of the triads in any pair are different from the corresponding elements of the other triad of the same pair is given by the expression $\mu\nu(\mu-1)(\nu-1)/2$. By allowing x to vary from 1 to λ, we have $\lambda\mu\nu(\mu-1)(\nu-1)/2$. Permuting λ, μ, ν and adding, we have the total number N' of the desired pairs of triads in the representation of F^n. Therefore, twice this number is

$$j=\lambda\mu\nu(\mu-1)(\nu-1)+(\nu-1)(\lambda-1)+(\lambda-1)(\mu-1).$$

To derive a formula for d, we consider a pair of triads with

* This formula can be obtained by the same method as that employed by Salmon to determine the rank of a 3-space curve which is the complete intersection of two surfaces. See Salmon, *Analytic Geometry of Three Dimensions*, 6th ed., vol. I, Paragraphs 342, 343.

corresponding elements all unlike. The planes they represent are all skew and there is just one line passing through a given point and incident with these planes. Such a line gives rise to an improper double point on the projection F'^n of F^n in S_4. Hence, the total number of pairs of triads in the representation of F^n of the nature just described is the number d. As each of the λ integral values of x is to be combined with each of the μ integral values of y and also with each of the ν integral values of z, we have

$$d = k\lambda\mu\nu(\lambda-1)(\mu-1)(\nu-1),$$

where k is a yet unknown constant. By actual trial for the case $\lambda=\mu=\nu=2$, we find $d=4$ and, therefore, $k=1/2$. Then

$$d = \tfrac{1}{2}\lambda\mu\nu(\lambda - 1)(\mu - 1)(\nu - 1)$$

is the required formula.

It is to be noted that if one of the three hypersurfaces, say V_4^ν, is a hyperplane S_4, that is, $\nu=1$, we have

$$d = 0, \quad j = \lambda\mu(\lambda - 1)(\mu - 1).$$

That is, the surface F'^n of order $n'=\lambda\mu$ which is the complete intersection of two hypersurfaces V_3^λ, V_3^μ, in S_4 cannot have improper double points and cannot be the projection of a surface of the same order in S_5. The number of its tangent lines passing through a given point in S_4 is always twice the order of the cone of lines passing through a given point and incident with the surface twice.

It remains to determine t. For this purpose we consider triples of triads in the representation of F^n. The triples that we need are of four types. Those of type I each represent three planes lying in a 4-space. Two corresponding elements of the triads of such a triple must be different, as (1,1,1), (1,2,2), (1,3,3) or (1,1,1), (3,1,2), (2,1,4). Those of type II each represent three planes lying two by two in three 4-spaces. Every pair of triads of such a triple must have one and only one element in common, as for example, (1,1,1), (1,2,2), (2,1,2) or (1,2,1), (1,4,3), (2,2,3) or (1,2,3), (1,4,2), (3,2,2). Type III consists of those triples each of which represents three planes such that two of them lie in a 4-space and the third may or may not lie in another 4-space with one of them. Two of the triads of

such a triple must have one element in common and the third may or may not have one of the remaining elements in common with one of them. Examples are (1,1,1), (1,2,2), (2,3,2); (1,1,1), (1,2,3), (2,3,2). Type IV consists of all those triples the triads of each of which have all their corresponding elements different, as (1,1,1), (2,2,2), (3,3,3) or (1,2,3), (2,3,1), (3,1,2). The planes represented by such a triple are all skew.

The three planes represented by any triple belonging to any of these four types are such that through a given line in S_5 not incident with them passes just one plane meeting them each in a point. Such a plane gives rise to an apparent triple point on the projection F'^n of F^n in S_4. Hence, to find t is to find the sum of the numbers T, T', T'', T''' of the triples of points belonging to the four types, respectively, obtained from the representation of F^n.

Reasoning in a manner analogous to that in which the formulas for a, j, d are derived, we obtain the following, which can be verified without difficulty:

$$\begin{aligned} T &= \tfrac{1}{6}\lambda\mu\nu[(\mu-1)(\mu-2)(\nu-1)(\nu-2) \\ &\quad + (\nu-1)(\nu-2)(\lambda-1)(\lambda-2) \\ &\quad + (\lambda-1)(\lambda-2)(\mu-1)(\mu-2)], \\ T' &= \lambda\mu\nu(\lambda-1)(\mu-1)(\nu-1), \\ T'' &= \tfrac{1}{2}\lambda\mu\nu(\lambda-1)(\mu-1)(\nu-1)(\mu\nu+\nu\lambda+\lambda\mu-12), \\ T''' &= \tfrac{1}{6}\lambda\mu\nu(\lambda-1)(\lambda-2)(\mu-1)(\mu-2)(\nu-1)(\nu-2). \end{aligned}$$

Then we have $t = T + T' + T'' + T'''$.

So far, we have dealt with surfaces which are complete intersections of hypersurfaces. It remains to say a few words concerning those surfaces which are partial intersections. For $n<\lambda\mu\nu$, we still have n triads in the symbolic representation of F^n. These n triads must be such that any of them has two corresponding elements in common with at least one other element. Every non-degenerate F^n has its own representation and every arrangement or group of n triads whose elements satisfy the above requirement represents a non-degenerate F^n in S_5. Given an F^n, we find the characteristics a, j, d, t of its projection F'^n in S_4 by counting the numbers of the respective pairs and triples of triads in its representation of the nature already explained. Let us illustrate.

A surface F^4 of order 4 in S_5 is of one of three types, represented respectively by the triads

	[1]	[2]	[3]	[4]
(a)	(1, 1, 1),	(2, 1, 1),	(1, 2, 1),	(1, 1, 2);
(b)	(1, 1, 1),	(1, 2, 1),	(1, 2, 2),	(2, 2, 2);
(c)	(1, 1, 1),	(1, 2, 1),	(1, 1, 2),	(1, 2, 2).

According to the principle explained above, it is not difficult to see that an F^4 represented by (a) is a Veronese quartic surface in S_5. Its projection F'^n in S_4 has one ($t=T'=1$) apparent triple point, for there is only one triple of triads given by [2], [3], [4] belonging to one of the four types explained above, indeed belonging to type II; and has no ($d=0$) improper double point, for there is no pair of triads with corresponding elements all unlike. The projection of F'^n in an S_3 is a Steiner's quartic surface and has one triple point and three double lines. As the number of pairs of triads with two corresponding elements different is 3, given by [2], [3]; [3], [4]; [2], [4], there are $j=2\cdot 3=6$ pinch-points on this projected surface in S^3; and as the number of pairs of triads with two corresponding elements alike is also 3, given by [1], [2]; [1], [3]; [1], [4], the tangent cone is also of order $a=2\cdot 3=6$.

Examining the four triads of (b) and those of (c) in a similar manner, we find that an F^4 represented by (b) has for projection in S_4 an F'^n for which $a=6$, $d=1$, $j=4$, and $t=0$. Its projection in S_3 is a ruled quartic surface with a double twisted cubic curve upon which lie four pinch-points and no triple point. We also find that the surface represented by (c) is already a 4-space surface, the Segre quartic surface, being the complete intersection of three hypersurfaces in S_5 one of which is a hyperplane and the other two are of order 2. The formulas above derived apply and we find $a=8$, $j=4$, $d=0$, $t=0$. Its projection in S_3 has a double conic on which are four pinch-points.

For other values of n we proceed in the same manner. Of course, when n is large, this process of counting is laborious but the desired characteristics can invariably be found.

THE UNIVERSITY OF CALIFORNIA

SINGULAR POINTS OF FUNCTIONS WHICH SATISFY THE PARTIAL DIFFERENTIAL EQUATION OF THE FLOW OF HEAT*

BY D. V. WIDDER

1. *Introduction.* It has frequently been pointed out that the function

$$U(x, y; a, b) = \frac{1}{(y-b)^{1/2}} e^{\mu}, \qquad (y > b),$$
$$= 0, \qquad (y \leqq b), \tag{1}$$

where $\mu = -(x-a)^2/[4(y-b)]$, plays a rôle in the theory of the solutions of the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \tag{2}$$

which is quite analogous to that played by the function

$$\log [(x-a)^2 + (y-b)^2]^{1/2} \tag{3}$$

in the theory of harmonic functions. One would expect, therefore, to find a characterization of the function (1) similar to that given by Bôcher† for (3). It is the purpose of the present note to obtain such a characterization.

For brevity we designate a function as regular in a given region if it is continuous with its first derivatives there. The results to be proved are the following.

THEOREM 1. *If $f(x, y)$ is a single-valued solution of* (2) *regular in the neighborhood of a point (a, b) except at (a, b), and is bounded, then $f(x, y)$ becomes regular at (a, b) if its definition at this point is properly adjusted.*

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† M. Bôcher, *Singular points of functions which satisfy partial differential equations of the elliptic type*, this Bulletin, vol. 9 (1903), p. 455. See also O. D. Kellogg, *On some theorems of Bôcher concerning isolated singular points of harmonic functions*, this Bulletin, vol. 32 (1926), p. 664. Professor Kellogg has informed the author that Bôcher's principal theorem was also known to Schwarz.

THEOREM 2. *If $f(x, y)$ is a single-valued solution of* (2) *regular in the neighborhood of a point (a, b) except at (a, b), and if $f(x, y)$ is bounded on one side only, then $f(x, y) - LU(x, y; a, b)$ is regular at (a, b) provided the constant L is properly chosen and the function is properly defined at (a, b).*

Our proof, like Bôcher's, employs a Green's function; but the method of proof must necessarily differ considerably from that of Bôcher since the level curves

$$\frac{1}{(y-b)^{1/2}}e^{\mu} = \text{constant}, \quad \mu = -\frac{(x-a)^2}{4(y-b)},$$

do not in this case consist of closed curves bounding regions with (a, b) in the interior, but all pass through this point.

2. *Green's Formula and Green's Function.* In the present section we shall recall certain results from the classical theory of equation (2). Set

$$F(u) = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}, \qquad G(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}.$$

Then $G(u)$ is the adjoint of $F(u)$ and the equation

$$G(u) = 0 \tag{4}$$

is the adjoint of equation (2). Let D be a regular* region with boundary C. If $u(x, y)$ and $v(x, y)$ are two functions continuous with their first and second derivatives in and on the boundary of D, then Green's formula is

$$\int_D \int [vF(u) - uG(v)]\,dx\,dy = \int_C uv\,dx + \left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)dy. \tag{5}$$

Here the contour integration is in the positive sense.† We now define the Green's function corresponding to a given rectangle.

DEFINITION. Let the vertices A, B, F, E of a rectangle R have

* For present purposes the boundary may consist of a finite number of straight-line segments.

† Most of the results of this section will be found in E. Goursat, *Cours d'Analyse*, 1923, vol. 3, Chap. 29. In applying this formula to regular solutions of (2) and (4) we make use of the fact that such solutions are known to have continuous second derivatives.

coördinates (x_0, y_0), (x_0+p, y_0), (x_0+p, y_0+q), (x_0, y_0+q) respectively, p and q being arbitrary positive constants, and let (ξ, η) be an arbitrary interior point. By the Green's function of this rectangle we mean a function $G(x, y; \xi, \eta)$ which, considered as a function of x and y:

(a) satisfies (2) and is regular in the closed region R except at (ξ, η);

(b) has the form

$$G(x, y; \xi, \eta) = U(x, y; \xi, \eta) + u(x, y)$$

at (ξ, η) where $u(x, y)$ satisfies (2) and is regular in the closed region R;

(c) is zero on the line segments EA, AB, BF.

The existence and uniqueness of this function follows from known results concerning a familiar boundary-value problem.* It may also be shown that G satisfies the adjoint equation (4) considered as a function of (ξ, η), and is regular except at the point (x, y), where it has the form

$$U(x, y; \xi, \eta) + v(\xi, \eta),$$

$v(\xi, \eta)$ being a solution of (4) regular throughout R. Moreover, G vanishes on the sides AE, EF, FB when considered as a function of (ξ, η).

3. *Three Lemmas.*

LEMMA 1. *If $\phi(x, t)$ is a solution of* (2) *regular in R except perhaps at an interior point (a, b), then the integral*

$$\int_C \frac{\partial \phi}{\partial x} dy + \phi dx$$

has a constant value K when extended in the clock-wise sense over the contours C of all rectangles lying in R and containing (a, b) as an interior point.

This follows from formula (5) by taking $u = \phi$, $v = 1$, and D the region between two of the rectangles considered. In particular, if ϕ is regular at (a, b), K is zero.

LEMMA 2. *If $\phi(x, y)$ satisfies the conditions of Lemma* 1, *is*

* E. Goursat, loc. cit., p. 316.

bounded at least on one side, and vanishes on the sides EA, AB, BF, then

$$\lim_{h\to 0^+} \int_{a-k}^{a+k} \phi(x, b+h) f(x)dx = Kf(a)$$

for any function $f(x)$ continuous in $(a-k, a+k)$ and for any k such that the points $(a-k, b)$, $(a+k, b)$ lie in R.

PROOF. Since $\phi(x, y)$ is bounded on one side it is possible to find a constant H such that the function $\psi(x, y) = \phi(x, y) + H$ is a function of one sign in R. For definiteness we suppose it positive. As a result of Lemma 1 we have

$$\int_{a-k}^{a+k} \phi(x, b+h)dx + \int_{b}^{b+k} \frac{\partial \phi}{\partial x}(a-k, y)dy + \int_{b+k}^{b} \frac{\partial \phi}{\partial x}(a+k, y)dy = K.$$

We have here chosen the contour C as the boundary of the rectangle whose vertices are $(a-k, b-h), (a+k, b-h), (a+k, b+h), (a-k, b+h)$. It was unnecessary to include the complete contour since ϕ vanishes identically for $y<b$. This follows* since ϕ is zero on the sides EA, AB, BF, and is regular for $y<b$. If h tends to zero we clearly have

$$\lim_{h\to 0^+} \int_{a-k}^{a+k} \phi(x, b+h)dx = K,$$

or

$$\lim_{h\to 0^+} \int_{a-k}^{a+k} \psi(x, b+h)dx = K + 2kH.$$

We shall first show that

$$\lim_{h\to 0^+} \int_{a-k}^{a+k} \psi(x, b+h) f(x)dx = Kf(a) + H \int_{a-k}^{a+k} f(x)dx.$$

The desired result will follow from this equation by noting that

$$\int_{a-k}^{a+k} \phi(x, b+h) f(x)dx = \int_{a-k}^{a+k} \psi(x, b+h) f(x)dx - H \int_{a-k}^{a+k} f(x)dx.$$

* E. Goursat, loc. cit., p. 309.

We employ the following identity:

$$
\begin{aligned}
I_h &= \int_{a-k}^{a+k} \psi(x, b+h) f(x)dx - Kf(a) - H\int_{a-k}^{a+k} f(x)dx \\
&= \int_{a-\alpha}^{a+\alpha} [f(x) - f(a)]\psi(x, b+h)dx + f(a)\int_{a-\alpha}^{a+\alpha} \psi(x, b+h)dx \\
&\quad - Kf(a) + \int_{a+\alpha}^{a+k} \phi(x, b+h) f(x)dx \\
&\quad + \int_{a-k}^{a-\alpha} \phi(x, b+h) f(x)dx - H\int_{a-\alpha}^{a+\alpha} f(x)dx.
\end{aligned} \tag{6}
$$

Here α is an arbitrary number for which $0<\alpha<k$. Given an arbitrary positive number ϵ, we determine α so small that the following inequalities hold:*

$$\left|\int_{a-\alpha}^{a+\alpha} f(x)dx\right| < \frac{\epsilon}{5H}, \tag{a}$$

$$|f(x) - f(a)| < \frac{\epsilon}{5(K+2kH)}, \quad |x-a| \leqq \alpha, \tag{b}$$

$$\alpha < \frac{\epsilon}{10H|f(a)|}. \tag{c}$$

With this choice of α equation (6) gives the inequality

$$
\begin{aligned}
|I_h| &\leqq \frac{\epsilon}{5(K+2kH)}\int_{a-\alpha}^{a+\alpha} \psi(x, b+h)dx \\
&\quad + |f(a)|\left|\int_{a-\alpha}^{a+\alpha} \psi(x, b+h)dx - K\right| \\
&\quad + \left|\int_{a+\alpha}^{a+k} \phi(x, b+h) f(x)dx\right| \\
&\quad + \left|\int_{a+k}^{a-\alpha} \phi(x, b+h) f(x)dx\right| + \frac{\epsilon}{5}.
\end{aligned} \tag{7}
$$

We can now choose h so small that the right-hand side of this

* H may clearly be chosen positive and so great that $K+2kH$ is positive. If $f(a)=0$, condition (c) becomes superfluous.

inequality will be less than ϵ. For we have shown that we can choose h so small that the integral

$$\int_{a-\alpha}^{a+\alpha} \psi(x, b + h)dx$$

differs as little as we like from $K+2\alpha H$, and hence is less than $K+2kH$. For such values of h the first term on the right-hand side of (7) is less than $\epsilon/5$. The second term may be made to differ as little as we like from $2\alpha H|f(a)|$ by choice of h, and hence made less than $\epsilon/5$ by virtue of (c). The two remaining integrals of (7) approach zero as h approaches zero since they are continuous functions of h in the neighborhood of $h=0$ and since $\phi(x, b)=0$ in the intervals $a+\alpha \leqq x \leqq a+k$, $a-k \leqq x \leqq a-\alpha$. We may therefore determine h so that each is in absolute value less than $\epsilon/5$. It follows that $|I_h| < \epsilon$ for h sufficiently small, and the lemma is established.

LEMMA 3. *If $\phi(x, y)$ satisfies the conditions of Lemmas* 1 *and* 2, *and if $v(x, y)$ is a solution of* (4) *regular in R, then*

$$(8) \quad \int_{C_0} \phi(x, y)v(x, y)dx + \left[v(x, y)\frac{\partial \phi}{\partial x}(x, y) - \phi(x, y)\frac{\partial v}{\partial x}(x, y)\right]dy = Kv(a, b),$$

where C_0 is the contour of any rectangle in R with its sides parallel to the axes and including (a, b) as an interior point, and where the integration is in the clockwise sense.

PROOF. Consider the function $\phi(x, y+h)$ where h is a small positive constant which, in the course of the proof, will be allowed to approach zero as its limit. This function is a solution of (2) regular in the rectangle whose vertices are $(a-k, b)$, $(a+k, b)$, $(a+k, b+r)$, $(a-k, b+r)$ provided r, h and k are chosen positive and sufficiently small; for, the point $(a, b-h)$ at which $\phi(x, y+h)$ may fail to be regular lies outside the rectangle. We may now apply Green's formula to this rectangle taking $u(x, y)=\phi(x, y+h)$ and $v(x, y)=v(x, y)$. Then

$$\int_{a-k}^{a+k} \phi(x, b + h)v(x, b)dx$$

$$= \int_b^{b+r} \left[v(a-k, y)\frac{\partial \phi}{\partial x}(a-k, y+h) - \phi(a-k, y+h)\frac{\partial v}{\partial x}(a-k, y) \right] dy$$

$$+ \int_{a-k}^{a+k} \phi(x, b+r+h)v(x, b+r)dx$$

$$+ \int_{b+r}^{b} \left[v(a+k, y)\frac{\partial \phi}{\partial x}(a+k, y+h) - \phi(a+k, y+h)\frac{\partial v}{\partial x}(a+k, y) \right] dy.$$

Now let h approach zero. By Lemma 2 the first integral tends to $Kv(a, b)$. The integrals on the right are continuous functions of h in the neighborhood of $h=0$, so that their limits are obtained by setting $h=0$ in their integrands.

It is now an easy matter to complete the proof. By Green's formula the integral (8) extended over C_0 is equal to the same integral extended over the rectangle whose vertices are $(a-k, b-r)$, $(a+k, b-r)$, $(a+k, b+r)$, $(a-k, b+r)$, where r and k are so chosen that the rectangle lies in R. But we have seen that $\phi(x, y) \equiv 0$ if $y<b$ so that the part of the contour for which $y<b$ may be neglected. The integral extended over the remainder of the contour, as we have just seen, is equal to $Kv(a, b)$. The equation (8) is thus established.

4. *Proof of the Theorems.* Let $f(x,y)$ be a solution of (2) regular in the neighborhood of (a, b) except at (a, b). Choose the rectangle R of Section 2 so that $f(x, y)$ is regular throughout R except at (a, b). Determine a solution $\bar{f}(x, y)$ of (2) regular throughout R and assuming the same values as $f(x, y)$ on the sides EA, AB, BF. If $f(x, y)$ is bounded at least on one side, then the function $\phi(x, y) = f(x, y) - \bar{f}(x, y)$ has the same property, and may be taken as the function ϕ of the lemmas.

Let (x_1, y_1) be an arbitrary interior point of R not the point (a, b). Surround (x_1, y_1) by a small rectangle with boundary C_1, with sides parallel to the axes, lying entirely in R, and such that (a, b) is an exterior point. In a similar way construct a rectangle with boundary C_0 about (a, b) and so small that all points of C_1 are exterior points. Let C be the boundary of R. Form the Green's function $G(x, y; \xi, \eta)$ for the region R. Fix the first pair of arguments at (x_1, y_1), and let the second pair be variable. The functions $\phi(x, y)$ and $G(x_1, y_1; x, y)$ are regular in the region D bounded by the curves C, C_0, C_1, the former a solution of (2), the latter a solution of (4). If we apply Green's formula to these two functions in the region D we obtain

$$\int_{C_0} \phi(x, y)G(x_1, y_1; x, y)dx + \left(G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right)dy$$
$$+ \int_{C_1} \phi(x, y)G(x_1, y_1; x, y)dx + \left(G\frac{\partial\phi}{\partial x} - \phi\frac{\partial G}{\partial x}\right)dy = 0.$$

The integration is in the same sense over C_0 and C_1. The integral extended over C disappears since either ϕ or G vanishes on each side of the rectangle. By Lemma 3, the first of these integrals has the value $\pm KG(x_1, y_1; a, b)$. The second may easily be evaluated by known theory. For, we recall that, in the neighborhood of (x_1, y_1), $G(x_1, y_1; x, y) = U(x_1, y_1; x, y) + v(x, y)$, where $v(x, y)$ is a solution of (4) regular in the region bounded by C_1. Moreover.

$$\int_{C_1} \phi v dx + \left(v\frac{\partial\phi}{\partial x} - \phi\frac{\partial v}{\partial x}\right)dy = 0,$$

and

$$\int_{C_1} \phi(x, y)U(x_1, y_1; x, y)dx + \left(-\phi\frac{\partial U}{\partial x} + U\frac{\partial\phi}{\partial x}\right)dy$$
$$= \pm 2\pi^{1/2}\phi(x_1, y_1).^{4}$$

Hence $2\pi^{1/2}\phi(x_1, y_1) = \pm KG(x_1, y_1; a, b)$. As a function of (x_1, y_1), G is a solution of (2), and we have

$$\phi(x, y) = \pm\frac{K}{2\pi^{1/2}}U(x, y; a, b) + u(x, y),$$

$$f(x, y) = \pm\frac{K}{2\pi^{1/2}}U(x, y; a, b) + u(x, y) + f(x, y). \tag{9}$$

We see that if $f(x, y)$ is bounded on both sides K must be zero, for, otherwise the right-hand side of (9) would be unbounded, and we should have a contradiction. On the other hand, if $f(x, y)$ is unbounded on one side, K can not be zero. If we choose the constant L of Theorem 2 as $\pm K/(2\pi^{1/2})$, equation (9) serves to prove that theorem. Both theorems are thus completely established.

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*E. Goursat, loc. cit., p. 311.

ON CERTAIN DOUBLY INFINITE SYSTEMS OF CURVES ON A SURFACE

BY C. H. ROWE

1. *Introduction.* A system of ∞^2 curves having been given on a surface, let us consider a variable curvilinear triangle bounded by three curves of the system. If we allow this triangle to shrink to a point P, the excess of the sum of its angles over π (or, briefly, its excess) will tend to zero and will in general be an infinitesimal of the same order as the perimeter, the ratio of the excess to the perimeter tending to a limit which depends not only on the position of the point P but also on the directions of the three curves through P with which the sides tend to coincide. For special systems, however, it may happen that the excess is an infinitesimal of the same order as the area, as in the case where the system consists of the geodesics of the surface; or it may happen that the limiting value of the ratio of the excess to the perimeter depends only on the position of the point P. We are thus led to ask whether limiting properties of this kind are characteristic of any interesting systems of ∞^2 curves on a surface; and we shall consider from this point of view velocity systems, systems of hypergeodesics, and the systems that have been studied by Ogura under the name of V-systems.

2. *Notation.* We shall use orthogonal curvilinear coordinates on the surface, the element of length being given by $ds^2 = Edu^2 + Gdv^2$. A rotation in the tangent plane will be called direct if it agrees in sense with the rotation through a right angle from the positive direction of the u-curve (v = constant) to the positive direction of the v-curve. When a positive sense has been defined along a curve on the surface, the direction that is obtained by giving a direct rotation of one right angle to the positive tangent will be called the positive normal to the curve in the tangent plane.

We shall denote by θ the inclination (measured in the direct sense) of the positive direction of the curve to the positive direction of the u-curve; and we shall denote by ρ the radius

of geodesic curvature of the curve with the usual convention of sign, so that ρ is positive if the center of geodesic curvature lies on the positive half of the normal in the tangent plane. The sign of ρ thus depends on our choice of the positive direction along the curve.

For a triangle bounded by three curves on the surface we shall denote the lengths of the sides by a_1, a_2, a_3, the perimeter $a_1+a_2+a_3$ by p, and the excess by e.

3. *Condition that e/p should Tend to Zero.* From the well known formula of Bonnet

$$\int d\theta - \int \frac{ds}{\rho} = \iint K dS,$$

in which K is the total curvature, we easily derive the expression

$$e = \int \frac{ds}{\rho} + \iint K dS \tag{1}$$

for the excess of a triangle bounded by three curves on the surface.* The curvilinear integral is taken around the boundary of the triangle in the direct sense, and this is the sense that we are to regard as the positive sense along the sides for the purpose of determining the value of the inclination θ and the sign of ρ.

We shall consider a triangle traced on the surface, and we shall allow it to shrink to a point P, so that the curves that form its sides tend to coincide with three curves through P having distinct tangents there. The directions that we agreed to regard as positive along the sides of the triangle determine in the limit certain directions along the three curves through P, and, regarding these directions as positive, we shall denote by θ_1, θ_2, θ_3 and by ρ_1, ρ_2, ρ_3 the values of θ and of ρ for these curves at P.

If we assume that the curves that form the sides tend in a sufficiently regular manner to their limiting positions, we may write the approximate formula

$$e = \sum \frac{a_r}{\rho_r} + o(p), \tag{2}$$

* Darboux, *Leçons sur la Théorie Générale des Surfaces*, vol. 3, p. 127.

where the summation extends over the values 1, 2, 3 of r, and where, in the usual notation, $o(p)$ denotes an error whose ratio to the perimeter tends to zero.

We shall now assume that the ratio of the excess to the perimeter tends to zero, so that

$$\sum \frac{a_r}{\rho_r} = o(p).$$

If we combine this with the formulas

$$\sum a_r \cos\theta_r = o(p), \qquad \sum a_r \sin\theta_r = o(p),$$

which we obtain by projecting the sides of the triangle on the tangents to the coordinate curves, we see that, whatever the numbers λ and μ are, we have

$$\sum a_r \left(\frac{1}{\rho_r} - \mu \cos\theta_r + \lambda \sin\theta_r \right) = o(p).$$

If we choose λ and μ so that two of the coefficients of a_1, a_2, a_3 in this sum are zero, the third must be zero also, because a_1, a_2, a_3 are infinitesimals of the same order as p. It follows that there exist numbers λ and μ such that the three equations

$$\frac{1}{\rho_r} = \mu \cos\theta_r - \lambda \sin\theta_r \qquad (r = 1, 2, 3) \tag{3}$$

are true simultaneously.

This shows that the centers of geodesic curvature of the three curves through P are collinear; for if we introduce rectangular cartesian coordinates in the tangent plane, taking the positive tangents to the u-curve and the v-curve as x-axis and y-axis respectively, the coordinates of these points are $-\rho_r \sin\theta_r$, $\rho_r \cos\theta_r$, and the equations (3) show that these points lie on the straight line whose equation is $\lambda x + \mu y = 1$.

The proof of the converse of this result presents no difficulty, and we may thus state the following theorem:

When a triangle traced on a surface shrinks to a point, a necessary and sufficient condition that the ratio of the excess to the perimeter should tend to zero is that the three curves with which the sides tend to coincide should have their centers of geodesic curvature at the point collinear.

We may notice a corollary which follows at once from this. *If three curves through a point have their centers of geodesic curvature collinear, they retain this property when the surface is transformed conformally.*

4. *Velocity Systems.* We shall now consider triangles bounded by curves belonging to a given system of ∞^2 curves on the surface, and we shall suppose that this system is determined by an equation of the form $1/\rho = f(\theta, u, v)$. It follows easily from (2) that, when the triangle shrinks to a point, e/p tends in general to a limit which depends on the position of this point and on the limiting directions of the sides of the triangle. We shall first ask what the nature of the system of curves must be in order that this limit should always be zero.

It will be clear at once from the theorem of the preceding paragraph that a necessary and sufficient condition for this is that the curves of the system that pass through any point should have their centers of geodesic curvature at the point collinear, or, in other words, that the equation that determines the system should be of the form

$$\frac{1}{\rho} = \mu(u, v) \cos\theta - \lambda(u, v) \sin\theta. \tag{4}$$

This condition means that the system of curves is a velocity system,* and we therefore have the following characteristic property of velocity systems:

A system of ∞^2 curves on a surface has the property that the excess of any infinitesimal triangle bounded by curves of the system is an infinitesimal of higher order than the perimeter if, and only if, the system is a velocity system.

It will be seen that this result leads immediately to the theorem that in a conformal transformation of the surface a velocity system is transformed into a velocity system.†

If in (1) we substitute the value of ρ given by (4) and write

$$\cos\theta\, ds = E^{1/2} du, \quad \sin\theta ds = G^{1/2} dv,$$

we obtain by means of Green's theorem the formula

* E. Kasner, *Natural families of trajectories: conservative fields of force*, Transactions of this Society, vol. 10 (1909), pp. 201–219.

† See Kasner, loc. cit., p. 126.

$$(5) \qquad e = \iint \left[K - \frac{1}{H} \frac{\partial}{\partial u}\left(\frac{\lambda H}{E^{1/2}}\right) - \frac{1}{H} \frac{\partial}{\partial v}\left(\frac{\mu H}{G^{1/2}}\right)\right] dS,$$

where

$$H^2 = EG.$$

We may therefore complete our result by adding that, *for an infinitesimal triangle formed with curves of a velocity system, the ratio of the excess to the area tends to a limit which depends only on the position of the point to which the triangle ultimately reduces.*

This limit is the integrand in the surface integral (5), and in order to interpret this expression we shall define a vector $\mathbf{F}$ in the tangent plane at each point P of the surface such that its projections on the positive tangents to the u-curve and the v-curve at P are λ and μ respectively. If we denote the surface divergence of this vector by div $\mathbf{F}$, it will be seen that we may write

$$\lim \frac{e}{A} = K - \operatorname{div} \mathbf{F},$$

where A is the area of the small triangle.

The vector $\mathbf{F}$ has the property that the geodesic curvature of any curve of the system at P is equal to the projection of $\mathbf{F}$ on the positive normal to the curve in the tangent plane.

If we recall the dynamical interpretation of a velocity system we may define the vector $\mathbf{F}$ in another way. The velocity curves on a surface relative to a given field of force and to a given speed v_0 have the property that at each of its points a velocity curve osculates a trajectory of a particle which moves on the surface under the action of the force and has at the point a velocity of magnitude v_0. It will be seen that the vector $\mathbf{F}$ associated with the velocity system is identical, except for a constant scalar factor, with the component of the force in the tangent plane.

We may notice that the condition that a velocity system should be an isogonal system is that e should vanish for every triangle, and therefore that the vector $\mathbf{F}$ should satisfy the equation div $\mathbf{F} = K$.* We may also remark that the velocity systems for which the vector satisfies the equation div $\mathbf{F} = 0$

* See Kasner, loc. cit., p. 217.

are the systems that have been mentioned by Darboux† as sharing with the geodesics of the surface the property expressed in the theorem of Gauss on the excess of a geodesic triangle.

5. *Hypergeodesics.* If we are given a system of ∞^2 curves which is not a velocity system, it will in general be possible to ensure that the excess of an infinitesimal triangle bounded by curves of the system is negligible compared with the perimeter by imposing a suitable restriction on the limiting directions of the sides. We may thus ask whether we can characterize any known systems of curves by the particular form that this restriction takes. We shall show that for the systems that have been called hypergeodesics by Wilczynski* this restriction takes a specially simple characteristic form.

A system of hypergeodesics consists of the curves that satisfy the differential equation obtained by equating d^2v/du^2 to a polynomial of the third degree in dv/du in which the coefficients are functions of u and v; and it is easily seen that this is equivalent to saying that the equation

$$\frac{1}{\rho} = f(\theta, u, v)$$

that determines the system is of the form

$$\frac{1}{\rho} = a\cos^3\theta + b\sin^3\theta + c\cos\theta + d\sin\theta, \tag{6}$$

where a, b, c, d are functions of u and v.

In order that the relation $e = o(p)$ should be satisfied by a triangle which is about to shrink to a point P, the centers of geodesic curvature at P of the curves of the system to which the sides tend must be collinear, and, with the notation of §3, the condition for this is

$$\frac{1}{\rho_1}\sin(\theta_2 - \theta_3) + \frac{1}{\rho_2}\sin(\theta_3 - \theta_1) + \frac{1}{\rho_3}\sin(\theta_1 - \theta_2) = 0.$$

If we replace the geodesic curvatures in this equation by their

† Darboux, *Sur une série de lignes analogues aux lignes géodésiques*, Annales de l'École Normale, (1), vol. 7 (1780), p. 176.

* E. J. Wilczynski, *Some generalizations of geodesics*, Transactions of this Society, vol. 23 (1922), pp. 223–239.

values as given by (6), we find after some simple reductions that our condition is equivalent to

$$\tan(\theta_1+\theta_2+\theta_3)=\frac{a}{b}, \tag{7}$$

provided that the three curves through P are distinct.

If we write

$$\bar{\theta}=\tfrac{1}{3}(\theta_1+\theta_2+\theta_3),$$

we may refer to the direction corresponding to the angle $\bar{\theta}$ as the *mean direction* of the sides of the infinitesimal triangle, and we notice that this direction depends only on the directions of the sides of the triangle and not on the direction of the u-curve from which the angles θ_r are measured. We may now say that, for a system of hypergeodesics, the condition that we have to impose on an infinitesimal triangle takes the form of a restriction placed on the mean direction of the sides; for the equation (7) requires that this direction should coincide with one of the six directions corresponding to the angles θ for which $\tan 3\theta=a/b$.

We shall next show that this special form of the condition is characteristic of systems of hypergeodesics. Consider a system of curves determined by an equation $1/\rho=f(\theta, u, v)$, which we may write $1/\rho=f(\theta)$ without risk of confusion, and we shall show that, if the equation

$$\theta_1+\theta_2+\theta_3=3k \quad \text{or} \quad \bar{\theta}=k,$$

where k is a given function of u and v, implies $e=o(p)$ or

$$f(\theta_1)\sin(\theta_2-\theta_3)+f(\theta_2)\sin(\theta_3-\theta_1)+f(\theta_3)\sin(\theta_1-\theta_2)=0,$$

then $f(\theta)$ has the form appropriate to a system of hypergeodesics.

We may assume without loss of generality that k is zero, because we can clearly arrange that this is so by a suitable choice of our coordinate system. The assumption that we are making about the function $f(\theta)$ is now equivalent to saying that it is a solution of the functional equation

$$f(\theta+\phi)\sin(\theta-\phi)+f(-\theta)\sin(\theta+2\phi)-f(-\phi)\sin(2\theta+\phi)=0. \tag{8}$$

It will be clear that, if $f(\theta)$ satisfies this equation, the function $g(\theta)$ defined by

$$g(\theta)=f(\theta)-(b\sin^3\theta+c\cos\theta+d\sin\theta)$$

does so also, b, c, d being any functions of u and v. We shall determine b, c, d so that $g(\theta)$ vanishes for the three values 0, α, β of θ, where α and β are two arbitrarily chosen numbers whose ratio is irrational.

If we write $\phi = -\theta$ in (8), we see that any solution of this functional equation that vanishes for $\theta = 0$ is an odd function. The function $g(\theta)$ is therefore an odd function, and the equation that it satisfies can consequently be written

$$g(\theta + \phi) \sin(\theta - \phi) = g(\theta) \sin(\theta + 2\phi) - g(\phi) \sin(2\theta + \phi).$$

Using this equation repeatedly, we see that the vanishing of $g(\alpha)$ and $g(\beta)$ implies that of $g(m\alpha + n\beta)$, where m and n are any two integers positive or negative. Since we can choose the integers m and n so that $m\alpha + n\beta$ is as close as we please to any given number, the function $g(\theta)$ must vanish identically, if we assume that it is continuous. The function $f(\theta)$ is therefore of the form (6) with $a = 0$, and the system of curves is accordingly a system of hypergeodesics. We may therefore state the following theorem:

When a system of ∞^2 curves on a surface is a system of hypergeodesics, but in no other case, we can ensure that the excess of an infinitesimal triangle bounded by curves of the system is negligible compared with the perimeter by imposing a restriction on the mean direction of the sides.

6. *V-systems.* We shall examine lastly the conditions under which infinitesimal triangles bounded by curves of a given system on a surface have the property that the ratio of the excess to the perimeter depends only on the position of the point to which the triangle ultimately reduces.

With the notation that we have used before, the condition that, as a triangle shrinks to a point, e/p should tend to a given limit η is that the inequality

$$\sum a_r \left(\frac{1}{\rho_r} - \eta \right) = o(p)$$

should be satisfied. Arguing as we did in §3, we see that this implies the existence of two numbers λ and μ such that the three equations

$$\frac{1}{\rho_r} - \eta = \mu \cos\theta_r - \lambda \sin\theta_r, \qquad (r = 1, 2, 3),$$

are true together. In order, therefore, that triangles formed from a given system of curves should have the limiting property in question, a relation of the type

$$\frac{1}{\rho} = \eta(u, v) + \mu(u, v) \cos\theta - \lambda(u, v) \sin\theta \tag{9}$$

must hold along the sides of every triangle.

The equation (9) shows that the system of curves must be such that the centers of geodesic curvature at any point P of the curves that pass through P lie on a conic one of whose foci is at P. Systems of this kind have been studied by Ogura* and have been named by him V-systems.

Moreover, the angle θ in the equation (9) corresponds to the direction in which the curve is described when a point moves in the direct sense around a triangle of which this curve forms a side, and, according to this equation, two distinct curves through a point correspond in general to the angles θ and $\theta+\pi$. This means that the equation indicates a certain sense along each curve and requires that, when the curve is used in forming a triangle, this sense should agree with the direct sense of description of the boundary of the triangle. We must consider only triangles for which this restriction is observed if the relation $e=p\eta(u, v)+o(p)$ is to be satisfied as the triangle shrinks to the point (u, v).

The restriction that we are imposing on the triangles may be expressed otherwise by saying that the equation (9) decides for us which half of the normal to the curve in the tangent plane is to point inwards when the curve forms part of the boundary of a triangle.

Conversely, if we are given a V-system, we can choose for each curve of the system the direction along the normal in the tangent plane that will have to point inwards when the curve is used in forming a triangle, and we can make this choice (at least for a limited portion of the surface) in such a way that continuity is observed as we move along a curve or pass from curve to curve. If this is done, it will be seen without difficulty that triangles formed with curves of the system have the limiting property under discussion.

* K. Ogura, *On certain systems of doubly infinite curves on a surface*, Tôhoku Mathematical Journal, vol. 8 (1915), pp. 213–217.

We may state our results in the following theorem:

V-systems, and no other systems, have the property that, for an infinitesimal triangle bounded by curves of the system, the ratio of the excess to the perimeter is a function of position, provided that we choose for each curve a direction along the normal in the tangent plane, having regard to continuity, and consider only triangles for which this direction points inwards.

We obtain as an immediate corollary to this a theorem given by Ogura which states that, when a surface is transformed conformally, a V-system is transformed into a V-system.

A simple illustration of our result is given by the V-system in the plane that consists of circles of constant radius a. The choice that we have to make amounts in this case to deciding whether the sides of a triangle shall have their convexities always turned outwards or always turned inwards, and, according as we adopt the first or the second of these alternatives, the excess of a triangle which conforms to our restriction is equal to p/a or to $-p/a$.

This example suggests the problem of finding the most general system of curves on a surface such that the excess of any triangle bounded by curves of the system is equal to a constant multiple of the perimeter, provided only that a certain restriction as to sense is observed. It will be seen that the solution is given by the V-system determined by an equation of the form (9) where $\eta(u, v)$ is a constant and where the associated velocity system defined by the equation

$$\frac{1}{\rho} = \mu(u, v) \cos\theta - \lambda(\quad v) \sin$$

is an isogonal system.

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THE CHARACTERISTIC ROOTS OF A MATRIX*

BY E. T. BROWNE

1. *Introduction.* If A is a square matrix of order n and I is the unit matrix, the equation in λ obtained by equating to zero the determinant $|A-\lambda I|$ is called the *characteristic equation* of A. The roots of this equation are called the *characteristic roots* of A. Although it is not possible to make any definite statement as to the nature of the characteristic roots of the general algebraic matrix A, several authors have given upper limits to the roots. The first upper limit seems to have been given by Bendixson† in 1900.

Let us denote by A' and $\overline{A}$ the transpose and the conjugate imaginary, respectively, of the square matrix A. If we write

$$B = \frac{A+\overline{A}'}{2}, \qquad C = \frac{A-\overline{A}'}{2i},$$

it is obvious that $\overline{B}'=B$ so that B is Hermitian (or real symmetric if A is real). Similarly, C is Hermitian (or skew-symmetric if A is real). Bendixson's theorem then is as follows:

BENDIXSON'S THEOREM. (a) *If $\alpha+i\beta$ is a characteristic root of a real matrix A and if $\rho_1 \geqq \rho_2 \geqq \cdots \geqq \rho_n$ are the characteristic roots (all real) of the symmetric matrix $B=(A+A')/2$, then*

$$\rho_1 \geqq \alpha \geqq \rho_n. \tag{1}$$

(b) *If g'' is the greatest of the numerical values of the elements $|(a_{ij}-a_{ji})/2|$ of the real skew-symmetric matrix $(A-A')/2$, then*

$$|\beta| \leqq g''[n(n-1)/2]^{1/2}. \tag{2}$$

The extension to the case where the elements of A are complex was made in 1902 by Hirsch‡ who proved the following theorem:

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* Presented to the Society, September 11, 1930.

† Bendixson, *Sur les racines d'une équation fondamentale*, Acta Mathematica, vol. 25 (1902), pp. 359–365.

‡ Hirsch, Acta Mathematica, vol. 25 (1902), pp. 367–370.

HIRSCH'S THEOREM. (a) *If $\alpha+i\beta$ is a characteristic root of any square matrix A and if we designate by g the greatest of $|a_{ij}|$,* by g' the greatest of $|(a_{ij}+\bar{a}_{ji})/2|$, and by g'' the greatest of $|(\bar{a}_{ij}-a_{ji})/2|$, then always*

$$|\alpha+i\beta| \leqq ng, \tag{3}$$

$$|\alpha| \leqq ng', \tag{4}$$

$$|\beta| \leqq ng''. \tag{5}$$

(b) *If ρ_1 is the greatest and ρ_n the least (algebraically) of the characteristic roots of $B=(A+\overline{A}')/2$, then always*

$$\rho_1 \geqq \alpha \geqq \rho_n. \tag{1'}$$

In 1904 Bromwich† gave a proof of Hirsch's Theorem (a) and (b) and further extended (b) as follows:

BROMWICH'S THEOREM. *If $\alpha+i\beta$ is a characteristic root of a matrix A and if we denote by $\pm\mu_1, \cdots, \pm\mu_v$ $(2v\leqq n)$ the non-zero characteristic roots of the matrix $C=(A-\overline{A}')/(2i)$, then $|\beta|$ cannot exceed the greatest of the $|\mu_i|$.*

In 1922, for a real matrix A, Pick‡ gave a proof of Bendixson's Theorem (a) with Bromwich's extension and he showed that (2) can be replaced by $|\beta|=g''|\operatorname{ctn}\pi/(2n)|$ which in general gives a more restricted limit than (2).

In 1927 the author§ attacked the problem from a different angle and proved the following theorem:

If λ is a characteristic root of a square matrix A and if $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all $\geqq 0$) of $A\overline{A}'$, then $\rho_1\geqq\lambda\bar{\lambda}\geqq\rho_n$.

It is the purpose of this paper to show by a very simple method that Hirsch's limits (3), (4) and (5) may be replaced

* Here $|A_{ij}|$ denotes not the determinant of the matrix A but the absolute value of the number A_{ij}.

† Bromwich, *On the roots of the characteristic equation of a linear substitution*, Acta Mathematica, vol. 30 (1906), pp. 295–304.

‡ Pick, *Über die Wurzeln der charakteristischen Gleichung von Schwingungsproblemen*, Zeitschrift für angewandte Mathematik und Mechanik, vol. 2 (1922), pp. 353–357.

§ *The characteristic equation of a matrix*, this Bulletin, vol. 34 (1928), pp. 363–368.

by limits which never exceed the former and are in general more restricted.

2. *The Characteristic Roots of* $A\bar{A}'$. If A is any square matrix, real or complex, Autonne* has shown that there exist two unitary (real orthogonal, if A is real) matrices P, Q such that

$$A = \bar{P}'NQ \tag{6}$$

where N has real positive (or zero) numbers in the main diagonal and zeros elsewhere. On forming the product $A\bar{A}' = \bar{P}'N^2P$ it becomes evident that the numbers in the diagonal of N are the positive (or zero) square roots $\rho_1^{1/2}, \cdots, \rho_n^{1/2}$ of the characteristic roots of $A\bar{A}'$. From (6), $N = PA\bar{Q}'$, so that if we denote by p_{ij}, n_{ij} and q_{ij} the element in the ith row and the jth column of P, N and Q, respectively, we have

$$n_{ij} = \sum_{r,s}^{1,\cdots,n} p_{ir}a_{rs}\bar{q}_{js}. \tag{7}$$

That is, the elements of N are of the form $\sum_{ij}^{1,\cdots,n} a_{ij}x_iy_j$ where $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$ are sets of numbers such that $\sum_i x_i\bar{x}_i = \sum_i y_i\bar{y}_i = 1$.

3. *An Upper Limit to the Roots of* A. Let us denote by η_i and ζ_i the absolute values of x_i and y_i, respectively. Then the sets $(\eta_1, \cdots, \eta_n)$ and $(\zeta_1, \cdots, \zeta_n)$ are real sets such that $\sum\eta_i^2 = \sum\zeta_i^2 = 1$. Since η_i and ζ_i are real

$$\eta_i\zeta_j \leqq \tfrac{1}{2}(\eta_i^2 + \zeta_j^2). \tag{8}$$

Hence

$$|n_{ii}| = |\sum a_{ij}x_iy_j| \leqq \sum |a_{ij}||x_i||y_j| = \sum |a_{ij}|\eta_i\zeta_j \leqq \tfrac{1}{2}\sum |a_{ij}|(\eta_i^2 + \zeta_j^2).$$

If S_i denotes the sum of the absolute values of the elements in the ith row of A and if S is the greatest of the S_i, we have

* Autonne, *Sur les matrices hypohermitiennes et les unitaires*, Comptes Rendus, vol. 156 (1913), pp. 858–860, in which the theorem is given without proof. In fact the theorem follows as a consequence of the author's Theorem IV on p. 367 of the aforementioned paper. See also Taber, *On the linear transformations between two quadrics*, Proceedings of the London Mathematical Society, vol. 24 (1892–93), pp. 290–306, in which the theorem for A real and non-singular is contained implicitly.

$$(9) \qquad \sum_{i,j} |a_{ij}| \eta_i^2 = \sum_i \eta_i^2 \sum_j |a_{ij}| = \sum_i \eta_i^2 S_i \leqq S \sum \eta_i^2 = S.$$

Similarly, if T_i denotes the sum of the absolute values of the elements in the ith column of A and if T is the greatest of the T_i,

$$\sum_{i,j} |a_{ij}| \zeta_j^2 \leqq T.$$

Hence

$$|n_{ii}| = \rho_i^{1/2} \leqq (S + T)/2.$$

Now from the author's theorem as quoted earlier in the paper, if λ is a characteristic root of A and M is the greatest of the ρ's, then

$$|\lambda| \leqq M^{1/2}.$$

Hence

$$|\lambda| \leqq (S + T)/2.$$

We therefore have the following theorem:

THEOREM. *If S_i (T_i) is the sum of the absolute values of the elements in the ith row (column) of a square matrix A and if S (T) is the greatest of the S_i (T_i), the absolute value $|\lambda|$ of a characteristic root λ of A cannot exceed $(S+T)/2$.*

Equation (3) of Hirsch's Theorem (a) obviously follows as a corollary to this theorem. It is clear that the limit given by the latter can never exceed that given by Hirsch's criterion and is in general less.

The limit is sometimes actually attained, for example, if A is a matrix each of whose elements is the square of the corresponding element of a real orthogonal matrix. In this case A obviously has the characteristic root $+1$. Moreover, the limit is always attained if A is a *circulant*, (that is, $a_{ij} = a_{j-i+1}$, for $i \leqq j$; $a_{ij} = a_{n+j-i+1}$ for $i > j$) whose elements are real and $\geqq 0$.

In particular, if A is Hermitian, $S_i = T_i$ and hence $S = T$, so that we have the following corollary.

COROLLARY 1. *If S_i is the sum of the absolute values of the elements in the ith row of an Hermitian matrix A, and if S is the greatest of the S_i, the numerical value of a characteristic root λ of A cannot exceed S.*

The limit given in the corollary is evidently attained if A consists entirely of zeros except in the diagonal. It is also attained if A is a real *cyclic* matrix, (that is, $a_{ij}=a_k$ where k is the least positive residue of $i+j-1$, mod n) with positive or zero elements.

On making use of Hirsch's Theorem (b) we have also the following corollary.

COROLLARY 2. *If $\alpha+i\beta$ is a characteristic root of A and if S_j' is the sum of the absolute values of the elements in the jth row of the Hermitian matrix $B=(A+\overline{A}')/2$, then if S' is the greatest of the S_j', it follows that $|\alpha|\leqq S'$.*

Equation (4) of Hirsch's Theorem follows directly from this corollary. Moreover, it is clear that our criterion usually gives a narrower limit than that of Hirsch.

By invoking Bromwich's Theorem we can state also the following corollary.

COROLLARY 3. *If $\alpha+i\beta$ is a characteristic root of A and if S_j'' is the sum of the absolute values of the elements in the jth row of the Hermitian matrix $C=(A-\overline{A}')/(2i)$, then if S'' is the greatest of the S_j'', we have $|\beta|\leqq S''$.*

Equation (5) of Hirsch's Theorem (a) follows directly from this corollary. Corollaries 2 and 3, which have been deduced from our theorem, might have been proved directly without invoking Autonne's theorem. For if we write, as in §1, $B=(A+\overline{A}')/2$, $C=(A-\overline{A}')/(2i)$, it is clear that $A=B+iC$ where B and C are Hermitian matrices. Suppose now that $\alpha+i\beta$ is a characteristic root of A. Then there exists a set $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$, and which we may suppose to have been divided through by the proper non-vanishing factor so that $\sum x_i\bar{x}_i=1$, such that

$$\sum_j a_{tj}x_j = \sum_j b_{tj}x_j + i\sum_j c_{tj}x_j = (\alpha + i\beta)x_t \quad (t = 1, \cdots, n).$$

On multiplying these equations through by $\bar{x}_t$ and summing as to t, we have

$$(10) \quad \sum_{t,j} b_{tj}\bar{x}_t x_j + i\sum_{t,j} c_{tj}\bar{x}_t x_j = \sum_t (\alpha + i\beta)\bar{x}_t x_t = \alpha + i\beta.$$

Now since B and C are Hermitian matrices each of the summations on the left is real. Hence, equating real and imaginary parts in (10) we have*

$$\alpha = \sum_{i,j} b_{ij}\bar{x}_i x_j,$$

$$\beta = \sum_{i,j} c_{ij}\bar{x}_i x_j.$$

If now we denote by η_i the absolute value of x_i and proceed as in (9), Corollaries 2 and 3 follow at once.

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ON THE REDUCTION OF THE INDEFINITE BINARY QUADRATIC FORMS†

BY J. V. USPENSKY

The reduction theory of the indefinite binary forms has been presented in widely different forms and from various points of view. But whatever point of view is adopted, it seems that Hermite's principle of continuous variables under more or less disguised form constitutes an essential foundation of all the existing theories of reduction.

Hermite's principle in its simplest aspect consists in association with a given indefinite form of a positive quadratic form containing a continuously varying positive parameter and the study of integral values of variables which give successive minima of the latter. However, the reduced forms in Hermite's theory differ from those in the classical Gaussian theory of reduction. A little contribution to the theory of reduction which this article contains has for its purpose to show how, by substituting for Hermite's positive quadratic form a certain non-homogeneous function containing a variable positive parameter, we obtain precisely the Gaussian reduced forms.

Let

$$\xi = \alpha x + \beta y, \quad \eta = \gamma x + \delta y$$

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* See Hirsch, loc. cit., p. 369.

† Presented to the Society, April 5, 1930.

be two independent linear forms. For the sake of simplicity we assume that the ratios $\alpha:\beta$ and $\gamma:\delta$ are irrational numbers so that neither of the forms ξ and η can vanish for integral values of the variables unless $x=y=0$. The function which takes the place of Hermite's positive form in the theory of reduction here developed is

$$F(x, y) = |\xi/\eta| + t|\eta|$$

where t is a continuously varying positive parameter. The variables x and y are supposed to be integers which do not vanish simultaneously. As $F(x, y)$ does not change if x and y are replaced by $-x$, $-y$ it is sufficient to consider such couples of integers (x, y) which make the second form η positive. It is almost evident that for a given positive value of $t=t_0$ there exists at least one couple of integers x, y giving a minimum value of $F(x, y)$. If such a couple is unique it will obviously be the only couple with such a property in a sufficiently small interval comprising the point $t=t_0$. If for $t=t_0$ there are several couples of integers giving the same minimum value of $F(x, y)$ we select one with the smallest η. This couple will give a minimum of $F(x, y)$ for $t=t_0$ and for all $t>t_0$ and sufficiently near to t_0. Thus there exist couples of integers which for certain values of the variable parameter are the only ones giving a minimum of $F(x, y)$. We shall call such couples of integers "minimizing couples." The whole question now is how to obtain all the minimizing couples.

Let (p_0, q_0) be one of the minimizing couples and τ the corresponding parameter value so that for $t=\tau$ the only system of integers giving a minimum of $F(x, y)$ is $x=p_0, y=q_0$ (provided the condition $\eta>0$ is satisfied). When t starts to decrease from τ the same couple which in a certain small neighborhood of τ continues to keep its property of being a minimizing couple cannot remain such in the whole interval from 0 to τ. For, given two numbers, ϵ however small and m however large, it is possible to find two integers x and y so as to satisfy the inequalities

$$|\xi| < \epsilon, \quad \eta > m$$

and it follows that corresponding to these integers $F(x, y)$ can be made less than any preassigned number if t becomes small enough whereas the same expression for $x=p_0$, $y=q_0$ remains

greater than a certain determined positive number however small t is. Hence, we can conclude the existence of a certain number $t_1 < \tau$ possessing the property that for $\tau \geqq t > t_1$ the couple p_0, q_0 is a unique minimizing couple, while for $t < t_1$ the same couple loses its minimizing property.

This can happen only in the following way: that for $t = t_1$ there are two minimizing couples; viz., p_0, q_0 and p_1, q_1, the latter taking the place of the former when t, decreasing, passes through the value t_1. We shall call this couple (p_1, q_1) the right neighbor of (p_0, q_0). If t begins to increase from τ in a similar manner we can see that the couple (p_0, q_0) ceases to give a minimum of $F(x, y)$ as soon as t passes through a certain value $t_0 > \tau$ and is replaced by another minimizing couple (p_{-1}, q_{-1}), its left neighbor.

So every minimizing couple possesses perfectly determined right and left neighbors, and all the existing minimizing couples can be arranged in a single chain:

$$\cdots, (p_{-2}, q_{-2}), (p_{-1}, q_{-1}), (p_0, q_0), (p_1, q_1), (p_2, q_2), \cdots$$

extending indefinitely in both directions. Of two consecutive couples in this series the following is the right neighbor of the preceding one and the latter is the left neighbor of the former. The chain of minimizing couples leads to a chain of linear substitutions

$$\begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix}; \qquad (i = \cdots, -2, -1, 0, 1, 2, \cdots).$$

We shall prove now the important property of this chain; namely, that all its substitutions are unimodular; that is, they have determinants ± 1. Let (p, q) and (p', q') be two minimizing couples, the second being the right neighbor of the first. It is evident that they both consist of relatively prime numbers. Furthermore, setting

$$\lambda = \alpha p + \beta q, \quad \lambda' = \alpha p' + \beta q',$$
$$\mu = \gamma p + \delta q, \quad \mu' = \gamma p' + \delta q',$$

we have by the properties of neighboring couples

$$\mu' > \mu > 0; \quad \left|\frac{\lambda'}{\mu'}\right| < \left|\frac{\lambda}{\mu}\right|; \tag{1}$$

and for suitable t

$$\left|\frac{\lambda}{\mu}\right| + t\,|\mu| = \left|\frac{\lambda'}{\mu'}\right| + t\,|\mu'| = P; \tag{2}$$

while for every couple of integers (x, y) differing from (p, q) and (p', q'), we have

$$\left|\frac{\xi}{\eta}\right| + t\,|\eta| \geqq P. \tag{3}$$

By means of the linear substitution

$$x = pX + p'Y, \quad y = qX + q'Y,$$

the forms ξ and η become

$$\xi = \lambda X + \lambda' Y, \quad \eta = \mu X + \mu' Y.$$

If the determinant has the value

$$pq' - p'q = \pm e, \; e > 1,$$

x and y will be integers if and only if we attribute to X, Y values of the form

$$X = \frac{a}{e}, \quad Y = \frac{b}{e}, \tag{4}$$

where a and b are integers satisfying the congruence

$$a + \mu b \equiv 0 \pmod{e}, \tag{5}$$

where μ is an integer determined by (p, q) and (p', q') and satisfying the conditions

$$0 < \mu \leqq e - 1.$$

If we substitute for X and Y their expressions (4), eliminate t by means of (2) and introduce the notations

$$\frac{\mu}{\mu'} = \sigma, \quad \frac{\lambda'}{\lambda} = \rho, \quad \rho\sigma = \epsilon,$$

the condition (3) can be presented as follows:

$$\frac{(1-\sigma)\,|a\sigma + b\epsilon|}{|a\sigma + b|} + e^{-1}(1 - |\epsilon|)\,|a\sigma + b| \geqq 1 - \sigma\,|\epsilon|; \tag{6}$$

while instead of (1) we have

$$0 < \sigma < 1, \ |\epsilon| < 1.$$

The inequality (6) must be satisfied for every couple of integers satisfying (5). In particular, it must be satisfied if we take

$$a = e - \mu \leqq e - 1, \ b = 1.$$

Setting for brevity

$$a\sigma = \zeta, \qquad 0 < \zeta \leqq (e-1)\sigma,$$

we must have

$$(7)\ (1-\sigma)|\zeta+\epsilon| + e^{-1}(1-|\epsilon|)(\zeta+1)^2 \geqq (\zeta+1)(1-\sigma|\epsilon|).$$

But

$$(1-\sigma)|\zeta+\epsilon| + e^{-1}(1-|\epsilon|)(\zeta+1)^2 \leqq (1-\sigma)(\zeta+|\epsilon|) + e^{-1}(1-|\epsilon|)(\zeta+1),$$

and

$$(1-\sigma)(\zeta+|\epsilon|) + e^{-1}(1-|\epsilon|)(\zeta+1)^2 - (1-\sigma|\epsilon|)(\zeta+1) = [1+\sigma\zeta - e^{-1}(\zeta+1)^2][|\epsilon|-1] < 0,$$

because in the interval $0 \leqq \zeta \leqq \sigma(e-1)$ the quadratic function

$$1 + \sigma\zeta - e^{-1}(\zeta+1)^2$$

remains positive provided $e > 1$. But the result obtained obviously contradicts (7), and hence $e = 1$. Instead of (6) we now have

$$(8)\ (1-\sigma)|a\sigma+\epsilon b| + (1-|\epsilon|)(a\sigma+b)^2 \geqq (1-\sigma|\epsilon|)|a\sigma+b|,$$

and this must hold for *every* couple of integers a and b. By taking $a=1, b=1$, and $a=1, b=-1$ we obtain two conditions

$$(9)\qquad (1-\sigma)|\sigma+\epsilon| + (\sigma+1)(\sigma-|\epsilon|) \geqq 0,$$

$$(10)\qquad |\sigma-\epsilon| + 2\sigma|\epsilon| - \sigma - |\epsilon| \geqq 0.$$

We can show now that the inequality $\sigma < |\epsilon|$ is impossible. For, if $\epsilon > 0$, we have

$$|\sigma-\epsilon| + 2\sigma|\epsilon| - \sigma - |\epsilon| = 2\sigma(\epsilon-1) < 0,$$

contrary to (10); and if $\epsilon < 0$ we have

$$(1-\sigma)|\sigma+\epsilon| + (\sigma+1)(\sigma-|\epsilon|) = 2\sigma(\sigma+\epsilon) < 0,$$

contrary to (9). Thus necessarily $|\epsilon| < \sigma$, the equality being impossible as we suppose the ratio $\beta:\alpha$ to represent an irrational number. Now, if $\epsilon > 0$, we have

$$|\sigma - \epsilon| + 2\sigma|\epsilon| - \sigma - |\epsilon| = 2\epsilon(\sigma - 1) < 0,$$

which again contradicts (10) and therefore $\epsilon < 0$, $-\epsilon < \sigma$; that is

$$-1 < \rho < 0, \quad 0 < \sigma < 1. \tag{11}$$

Thus we reach the following conclusion. If (p, q) is a minimizing couple and (p', q') its right neighbor, then

$$pq' - p'q = \pm 1,$$

and

$$-1 < \rho < 0, \quad 0 < \sigma < 1.$$

It can be shown conversely that, if these conditions are fulfilled, (p, q) and (p', q') are both minimizing couples and (p', q') right neighbor of (p, q). To this end, it suffices to show that the inequality (8) holds *with the excluded sign of equality* for every couple of integers (a, b) different from $(\pm 1, 0)$ and $(0, \pm 1)$. Now that we know the sign of $\epsilon = \sigma\rho$, we can write this inequality as follows:

$$(\sigma - \sigma^2)|a + b\rho| + (1 + \rho\sigma)(a\sigma + b)^2 - (1 + \rho\sigma^2)|a\sigma + b| \geqq 0. \tag{12}$$

Suppose at first that $a + b\rho < 0$, which necesitates $b > a$, if we suppose $a > 0$, which is legitimate. In this case, the left member of (12) coincides with the linear function ρ

$$P(\rho) = -(\sigma - \sigma^2)(a + b\rho) + (1 + \rho\sigma)(a\sigma + b)^2 - (1 + \rho\sigma^2)(a\sigma + b),$$

and for extreme values of ρ:

$$\rho = -1 \quad \text{and} \quad \rho = -ab^{-1},$$

we have

$$P(-1) = (1 - \sigma)[(a^2 - a)\sigma^2 + 2a(b - 1)\sigma + b^2 - b] \geqq 0,$$

$$P(-ab^{-1}) = b^{-1}(b + \sigma a)[b^2 - b - \sigma^2(a^2 - a)] > 0,$$

so that $P(\rho) > 0$. Suppose now $a + b\rho > 0$ and $a\sigma + b < 0$, which necessitates $b < 0$. In this case the left hand member of (12) coincides with the linear function of ρ

$$P(\rho) = (\sigma - \sigma^2)(a + b\rho) + (1 + \rho\sigma)(a\sigma + b)^2 + (1 + \rho\sigma^2)(a\sigma + b).$$

Now for $\rho = 0$ and $\rho = -1$, we have

$$P(0) = a(\sigma - \sigma^2) + (a\sigma + b)^2 + a\sigma + b,$$
$$P(-1) = (1 - \sigma)[\sigma(a - b) + (a\sigma + b)^2 + (1 + \sigma)(a\sigma + b)].$$

Since

$$\sigma(a - b) + (a\sigma + b)^2 + (1 + \sigma)(a\sigma + b) - [a(\sigma - \sigma^2) + (a\sigma + b)^2 + a\sigma + b] = 2a\sigma^2 > 0,$$

it suffices to consider $P(0)$. We have then

$$P(0) = (a^2 - a)\sigma^2 + 2a(b + 1)\sigma + b^2 + b.$$

The minimum of this function is attained for

$$\sigma_0 = -(b + 1)(a - 1)^{-1},$$

and is

$$-(b + 1)(a - 1)^{-1}(a + b).$$

If $\sigma_0 \leqq 1$, we have $a + b \geqq 0$, and this minimum is $\geqq 0$. But for $\sigma = 0$ and $\sigma = 1$, we have

$$P(0) = b^2 + b \geqq 0, \quad P(0) = (a + b)^2 + (a + b) \geqq 0.$$

Therefore, for $0 < \sigma < 1$, we have $P(0) \geqq 0$, $P(-1) > 0$; and hence

$$P(\rho) > 0.$$

Next, if $a\sigma + b > 0$, the left member of (12) coincides with

$$P(\rho) = (\sigma - \sigma^2)(a + b\rho) + (1 + \rho\sigma)(a\sigma + b)^2 - (1 + \rho\sigma^2)(a\sigma + b).$$

Again

$$P(0) = a(\sigma - \sigma^2) + (a\sigma + b)^2 - (a\sigma + b),$$
$$P(-1) = (1 - \sigma)[\sigma(a - b) + (a\sigma + b)^2 - (1 + \sigma)(a\sigma + b)],$$

and

$$\sigma(a - b) + (a\sigma + b)^2 - (1 + \sigma)(a\sigma + b) - [a(\sigma - \sigma^2) + (a\sigma + b)^2 - (a\sigma + b)] = -2b\sigma;$$

and this is > 0 or < 0 according as $b < 0$ or $b > 0$. If $b < 0$, we consider

$$P(0) = (a^2 - a)\sigma^2 + 2ab\sigma + b^2 - b.$$

For $\sigma=0$ and $\sigma=1$, this function assumes the values

$$b^2 - b > 0, \quad (a+b)^2 - (a+b) \geqq 0.$$

Its minimum is attained for

$$\sigma_0 = -b(a-1)^{-1},$$

and is

$$-b(a-1)^{-1}(a+b-1).$$

Now if $\sigma_0 \leqq 1$, we have

$$a+b-1 \geqq 0,$$

so that the minimum is $\geqq 0$. Therefore $P(0) \geqq 0$, if $0<\sigma<1$, and $P(-1)>0$; whence again

$$P(\rho) > 0.$$

Finally, if $b>0$, we consider the quadratic function

$$\sigma(a-b) + (a\sigma+b)^2 - (1+\sigma)(a\sigma+b)$$
$$= (a^2-a)\sigma^2 + 2b(a-1)\sigma + b^2 - b \geqq 0,$$

so that $P(-1) \geqq 0$ and $P(0)>0$, if $0<\sigma<1$. Hence $P(\rho)>0$. Thus the inequality (12) holds with the sign $>$ for every couple of integers (a, b) different from $(\pm 1, 0)$ and $(0, \pm 1)$; and this completes the proof of the statement.

Summing up, we can say that the conditions

$$pq' - p'q = \pm 1, \quad 0 < \sigma < 1, \quad -1 < \rho < 0,$$

are necessary and sufficient conditions in order that couples (p, q) and (p', q') should be two consecutive minimizing couples, the second being the right neighbor of the first. If we disregard the condition $\eta>0$, but instead suppose that the determinants of the substitutions in the chain belonging to the forms (ξ, η) have the value 1, then the preceding conditions are

$$pq' - p'q = 1, \quad |\sigma| < 1, \quad |\rho| < 1,$$

and σ and ρ have opposite signs.

After these preliminary investigations the reduction theory of the indefinite binary quadratic forms can be summarized in a few words.

Let

$$\phi = ax^2 + 2bxy + cy^2$$

be an indefinite form whose determinant $\mathfrak{D}>0$ is not a perfect square. Denoting by

$$\omega = \frac{-b+\sqrt{\mathfrak{D}}}{a}, \qquad \omega' = \frac{-b-\sqrt{\mathfrak{D}}}{a}$$

the first and the second roots of ϕ, we introduce the linear forms

$$\xi = x - \omega y, \quad \eta = x - \omega' y,$$

and can form the chain of substitutions belonging to (ξ, η) or simply to ϕ. We call ϕ a *reduced form* if its chain contains the identical substitution

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Necessary and sufficient conditions for this are that we should have $|\omega| < 1$, $|\omega'| > 1$, and that ω and ω' of opposite signs. In this characteristic we recognize the Gaussian reduced forms. Every form Φ is equivalent to a reduced form. For if

$$\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$$

is a substitution belonging to the chain of Φ and we apply this substitution to Φ we obtain a form ϕ whose chain contains the identical substitution and therefore ϕ is a reduced form.

Finally, the fundamental proposition that two forms Φ and Φ' cannot be equivalent unless they have the same chain of reduced forms is almost intuitive from the adopted point of view. However, the same is true of every reduction theory based on Hermite's principle of continuous variables.

STANFORD UNIVERSITY

CONCERNING A SET OF AXIOMS FOR THE SEMI-QUADRATIC GEOMETRY OF A THREE-SPACE*

BY J. L. DORROH

In his paper *Sets of metrical hypotheses for geometry*,† R. L. Moore raises the question whether the set O of order axioms and the set C of congruence axioms employed therein, together with M, the proposition that every segment has a mid-point, and P_2, a form of the parallel postulate, are sufficient to give the semi-quadratic geometry of a three-space. At the same time, he states that this question may be answered in the affirmative if it can be proved on the basis of O, C, and M that all right angles in space are congruent to each other. In the present paper it will be shown that O and C are sufficient to require that all right angles in space be congruent to each other.

It is a result of a recent paper‡ of the present author that the theorems of sections 1, 2, 3, and 4 of M.H. are consequences of O and C. Theorems from these sections of M.H. will be quoted without further mention of this justification of their use.

THEOREM 1. *If A, B, C, D are four non-coplanar points such that $\sphericalangle ABD$ is a right angle§ and $\sphericalangle CBD$ is a right angle, and E is any point distinct from B and in the plane ABC, then $\sphericalangle EBD$ is a right angle.*

PROOF. If E is a point of the line AB, or of the line CB, then, by hypothesis, $\sphericalangle EBD$ is a right angle.

Suppose, then, that E belongs to the plane ABC, is distinct from B, and belongs neither to the line AB nor to the line BC. Let C' denote a point such that CBC'. It follows by a corollary

* Presented to the Society, September 6, 1928.

† Transactions of this Society, vol. 9 (1908), pp. 487–512. The notation M. H. will be used to designate this paper. Similarly, S. A. will be used to denote O. Veblen's paper, *A system of axioms for geometry*, ibid., vol. 5 (1904), pp. 343–384.

‡ *Concerning a set of metrical hypotheses for geometry*, Annals of Mathematics, (2), vol. 29 (1928), pp. 229–231.

§ See Definition 7 of M. H., §3.

of Theorem 16 of S.A. that the line BE contains a point H such that AHC or AHC'. Let G denote one of the points C or C' so that AHG. Let F denote a point such that DBF and $DB \equiv BF$. Since by hypothesis the line BD is perpendicular to the line AB and to the line BG, it follows that $DG \equiv FG$ and $AD \equiv AF$. Since $AG \equiv AG$ and $AH \equiv AH$, it follows* that $DH \equiv FH$. Hence, by definition, $\measuredangle DBH$ is a right angle.

THEOREM 2. *If L, M, N, O are four non-coplanar points such that $\measuredangle LON$ is a right angle and $\measuredangle MON$ is a right angle, then $\measuredangle LON \equiv \measuredangle MON$.*

PROOF. Since L, M, N, O are non-coplanar, L, O, M are non-collinear. Let E denote a point such that the ray OE bisects $\measuredangle LOM$.† Let M' denote a point in the order MOM', and let Q denote a point such that the ray OQ bisects $\measuredangle M'OL$. Then $\measuredangle EOQ$ is a right angle.‡ Let P denote a point such that QOP and $OP \equiv OQ$; then $QE \equiv PE$. Also, since by Theorem 1 $\measuredangle NOP \equiv \measuredangle NOQ$, $QN \equiv PN$. The ray OM contains a point K such that PKE, and the ray OL contains a point R such that QRE. By Theorem 1 of M.H. §3, $OK \equiv OR$ and $EK \equiv ER$. Since PKE, QRE, $EP \equiv EQ$, $NE \equiv NE$, $NP \equiv NQ$, and $EK \equiv ER$, then $NK \equiv NR$,§ and, by definition, $\measuredangle NOR \equiv \measuredangle NOK$.

THEOREM 3. *If α_1 and α_2 are two intersecting planes and ϕ_1 is a right angle in α_1 and ϕ_2 is a right angle in α_2, then $\phi_1 \equiv \phi_2$.*

PROOF. Let k denote the line of intersection‖ of α_1 and α_2. Let k_1 denote a line in α_1 perpendicular to k at a point O of k, and let k_2 denote a line in α_2 perpendicular to k at O. Let ψ_1 be a right angle formed by k_1 and k, and let ψ_2 be a right angle formed by k_2 and k. It follows from Theorem 2 that $\psi_1 \equiv \psi_2$.

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* A special case of Theorem 11 of M. H. §1 may be stated as follows: *If A, B, C are three non-collinear points and A', B', C' are three non-collinear points, and ADC, $A'D'C'$, $AB \equiv A'B'$, $AC \equiv A'C'$, $AD \equiv A'D'$, $BC \equiv B'C'$, then $BD \equiv B'D'$.* For the suggestion that the figure used in the proof of Theorem 1 and the use of the particular theorem just stated would shorten the arguments I had previously given for Theorems 1 and 2, I am indebted to H. G. Forder.

† See a corollary of Theorem 6 of M. H., §3.

‡ See proof of Theorem 7 of M. H., §3.

§ See the theorem stated in a footnote on Theorem 1.

‖ See Theorem 25 of S. A., p. 363.

By Theorem 1 of M.H. §4, $\phi_1 \equiv \psi_1$, and $\phi_2 \equiv \psi_2$. It follows, then, from Theorem 14 of M.H. §1, that $\phi_1 \equiv \phi_2$.

THEOREM 4. *If ϕ_1 and ϕ_2 are two right angles in space, then $\phi_1 \equiv \phi_2$.*

PROOF. If ϕ_1 and ϕ_2 are in the same plane, $\phi_1 \equiv \phi_2$ by Theorem 1 of M.H. §4. If ϕ_1 and ϕ_2 are not in the same plane, they lie in intersecting planes or in non-intersecting planes. If they lie in intersecting planes, they are congruent to each other by Theorem 3. If ϕ_1 and ϕ_2 lie in the planes α_1 and α_2, respectively, and α_1 does not intersect α_2, there exists a plane α_3 which intersects both α_1 and α_2. There exists in α_3 a right angle ϕ_3. By Theorem 3, $\phi_1 \equiv \phi_3$ and $\phi_2 \equiv \phi_3$; hence, by Theorem 14 of M.H §1, we have $\phi_1 \equiv \phi_2$.

THE UNIVERSITY OF TEXAS

CERTAIN QUINARY FORMS RELATED TO THE SUM OF FIVE SQUARES*

BY B. W. JONES†

1. *Introduction.* The number of solutions in integers x, y, z of the equation $n = x^2+y^2+z^2$ is a function of the binary class number of n. For numerous forms $f = ax^2+by^2+cz^2$, the expression of the number of solutions of $f = n$ in terms of the class number is another way of showing that the number of representations of n by f is a function of the number of representations of various multiples of n as the sum of three squares.‡

Similarly, the number of solutions of the equation $n = x^2+y^2+z^2+t^2$ in integers is the sum of the positive odd divisors of n, multiplied by 8 or 24, according as n is odd or even. There are various forms $f = ax^2+by^2+cz^2+dt^2$ for which the number of representations of n by f is a multiple of the sum of the odd divisors of n. The number of representations of n by one of

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* Presented to the Society, April 5, 1930.

† National Research Fellow.

‡ See, for example, Kronecker, Journal für Mathematik, vol. 57 (1860), p. 253; J. V. Uspensky, American Journal of Mathematics, vol. 51 (1929), p. 51; B. W. Jones, American Mathematical Monthly, vol. 36 (1929), p. 73.

these forms is thus a simple function of the number of representations of n as the sum of four squares.*

Following a suggestion of E. T. Bell, I have here considered as the fundamental function, $\phi(n)$, the number of representations of n as the sum of five squares. With the exception of three forms (f_{11}, f_{12}, f_{13}), the number of solutions of $n=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2$, where $a_i=1$, 2 or 4, is shown to be expressible in terms of ϕ and for $a_i=1$, 2, 4, or 8 the number of solutions of $f=n$ is expressed in terms of ϕ and two other functions (α and β). It should be noted that for certain values of n, $M_{abc}(n)$ is expressible *totally* in terms of ϕ. It is true in many cases when n is even and in the following formulas when n is odd: (37.2), (37.3), (38.1), (40.1), (40.2), (41.1), (44.1), (46.1), (48.1), (48.2), (50.1), (52.1). See also the last section of the paper giving a few miscellaneous results.

2. *Notations.* The letters n, m, x, y, μ are used to denote integers; m and μ are odd and n and m are positive.

$N[n=f]$ denotes the number of representations of n by the form $f=x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2$, the coefficients to be arranged in *increasing order of magnitude.*

f_j or f_j' is the form f when j of the coefficients are 2 or 4 respectively and the rest of the coefficients are 1.

f_{ij} is the form f when i of the coefficients are 2, j of them 4 and the remainder are 1.

f_{abc} is the form $x_1^2+ax_2^2+bx_3^2+cx_4^2+8x_5^2$ where a, b and c are powers of 2.

$M_j(n)=N[n=f_j]$; $M_j'(n)=N[n=f_j']$; $M_{ij}(n)=N[n=f_{ij}]$; $M_{abc}(n)=N[n=f_{abc}]$.

We regard the following as fundamental functions:
$M_0(n)=\phi(n)$; $\alpha(m)=M_{11}(m)$; $\beta(m)=M_{244}(m)$, if $m\equiv 1 \pmod 8$.

We also use the following for brevity's sake: $\lambda(n)=M_1'(n)$; $\lambda'(4n)=N[4n=f_1'$; $x_1x_2x_3x_4$ odd$]$; $\alpha'(n)=N[n=f_{11}$ with x_1 odd$]$ and $\phi'(m)=N[m=f_0$; $x_1x_2x_3x_4x_5$ odd$]$, which has a value different from 0 only when $m\equiv 5 \pmod 8$.

3. *A Fundamental Lemma.*† $N[2n=x^2+y^2]=N[n=x^2+y^2]$.

* See, for example, J. Liouville, Journal de Mathématiques, (2), vol. 7 (1862); P. Pepin, Journal de Mathématiques, (4), vol. 6 (1890), p. 5.

† Since this lemma is very elementary, we shall use it freely without comment.

This follows from the fact that $2n=x^2+y^2$ implies that the pair of equations $x+y=2X$ and $x-y=2Y$ is solvable for X and Y and there is a one to one correspondence between the solutions of $2n=x^2+y^2$ and $n=X^2+Y^2$.

COROLLARY 1.

$$N[2m = x^2 + y^2] = 2N[m = \mu^2 + 4y^2].$$

COROLLARY 2.

$$N[2n' = x^2 + y^2] = N[n' = x^2 + y^2] = N[2n' = 2x^2 + 2y^2].$$

4. *Reduction Formulas for* $\phi(n)$. Since $f_0=n\equiv 0 \pmod 4$ implies that just one or all of the x's are even, we have

$$\phi(4n) - \phi(n) = 5\lambda'(4n). \tag{1}$$

Applying Corollary 1, we have

$$\begin{aligned}\lambda'(4n) &= 4N[2n = \mu_1^2 + \mu_2^2 + 4x_3^2 + 4x_4^2 + 2x_5^2] \\ &= 8N[n = \mu_1^2 + 4x_2^2 + 2x_3^2 + 2x_4^2 + x_5^2].\end{aligned}$$

Applying Corollary 2, we have

$$\lambda'(4n) = 8N[n = \mu_1^2 + 4x_2^2 + x^2 + y^2 + x_5^2;\ x \equiv y \pmod 2], \tag{1'}$$

$$\lambda'(4n) = 8\lambda'(n) \text{ if } n \equiv 0 \pmod 4. \tag{2}$$

If $n=2m$ note that $f_0=2m$ implies that exactly two x's are odd and we have from (1')

$$\lambda'(8m) = 4\phi(2m)/5. \tag{3}$$

If $n=m$ consider first the case $m\equiv 5 \pmod 8$. Then

$$\begin{aligned}\phi'(m) &= 4N[m = f_{22s} \text{ with } x_1x_2x_3 \text{ odd}] \\ &= 8N[m = \mu_1^2 + 4\mu_2^2 + 8x_3^2 + 16x_4^2 + 8x_5^2].\end{aligned}$$

Now $f_4'=m$ implies that one of x_2, x_3, x_4, x_5 is incongruent mod 2 to the other three, and thus

$$\begin{aligned}M_4'(m) &= 4N[m = f_4';\ x_2 \not\equiv x_3 \equiv x_4 \equiv x_5 \pmod 2] \\ &= 4N[m = \mu_1^2 + 4\mu_2^2 + 16x_3^2 + 8x_4^2 + 8x_5^2;\ x_4 \equiv x_5 \pmod 2] \\ &\quad + 4N[m = \mu_1^2 + 16x_2^2 + 4\mu_3^2 + 8x_4^2 + 8x_5^2;\ x_4 \not\equiv x_5 \pmod 2] \\ &= 4N[m = \mu_1^2 + 4\mu_2^2 + 8x_3^2 + 16x_4^2 + 8x_5^2].\end{aligned}$$

Thus $M_4'(m)=\phi'(m)/2$. This taken with the known equation

$5M_4'(m)+\phi'(m)=\phi(m)$, found by noting that $f_0=m$ implies that just one or all the x's are odd, gives

$$\phi'(m)2\phi(m)/7 \quad \text{if} \quad m \equiv 5 \pmod 8, \tag{4}$$

and

$$M_4'(m) = \phi(m)/7 \quad \text{if} \quad m \equiv 5 \pmod 8. \tag{5}$$

Now from (1'), we have $\lambda'(4m)=8M_4'(m)$ if $m\equiv 5 \pmod 8$. Since $f_0=m$ implies that just three of the x's are odd, or just one is odd according as $m\equiv 3 \pmod 4$ or $\equiv 1 \pmod 8$, we have, from (1') and (5),

$$\lambda'(4m) = 8a\phi(m), \tag{6}$$

where $a=1/10$, $1/5$, or $1/7$, according as

$$m \equiv 3 \pmod 4, \equiv 1 \pmod 8, \quad \text{or} \quad \equiv 5 \pmod 8.$$

If $n' \not\equiv 0 \pmod 4$, we have, using (1) and (2),

$$\phi(4^\alpha n') - \phi(4^{\alpha-1}n') = 5\cdot 8^{\alpha-1}\lambda'(4n'), \ (\alpha \geqq 1),$$
$$\phi(4^{\alpha-1}n') - \phi(4^{\alpha-2}n') = 5\cdot 8^{\alpha-2}\lambda'(4n'), \ (\alpha \geqq 2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\phi(4n') - \phi(n') = 5\lambda'(4n').$$

Adding, we get

$$\phi(4^\alpha n') - \phi(n') = 5\frac{8^\alpha - 1}{7}\lambda'(4n'), \tag{7}$$

where $\alpha \geqq 1$ and $n_3 \not\equiv 0 \pmod 4$. Then, using (3) and (6), we have the reduction formulas

$$\phi(2^{2\alpha+1}m) = (2^{3\alpha+2} + 3)\phi(2m)/7, \tag{8.1}$$

for $\alpha \geqq 1$,

$$\phi(4^\alpha m) = A\phi(m)/7, \tag{8.2}$$

where $\alpha \geqq 1$ and $A = 2^{3\alpha+2}+3$, $8^{\alpha+1}-1$, $(5\cdot 8^{\alpha+2}+9)/7$ according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

5. *Relationship between* $\lambda(n)$ *and* $\phi(n)$. It is obvious that $\lambda'(4n)=\lambda(4n)-\phi(n)$. Thus, from (1), we have $\phi(4n)+4\phi(n)=5\lambda(4n)$. This from (8) gives the formulas below for $\lambda(4n)$. We

find $\lambda(m)$ by noting that $f_0=m$ implies that just three of the x's are odd or just one is odd according as $m\equiv 3 \pmod 4$ or $m\equiv 1 \pmod 8$ and that $\lambda(m)=4M_4'(m)$ if $m\equiv 5 \pmod 8$. To obtain $\lambda(2m)$ note that $f_0=2m$ implies just two x's are odd.

$$\lambda(4^{\alpha+1}m) = B\phi(m), \tag{9.1}$$

where $\alpha\geqq 0$ and $B=3(2^{3\alpha+4}+5)/35$, $(3\cdot 2^{3\alpha+5}-5)/35$, or $3(2^{3\alpha+5}+3)/49$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

$$\lambda(4^{\alpha}\cdot 2m) = 3(2^{3\alpha+1}+5)\phi(2m)/35, \tag{9.2}$$

where $\alpha\geqq 0$;

$$\lambda(m) = 4a\phi(m), \tag{9.3}$$

where a is defined in (6).

6. *Forms f, where $M(n)$ is Expressible Totally in Terms of ϕ.**

CASE I: $n=m$. Note that $\lambda(2m)=6N[f_1'=2m;\ x_1x_2 \text{ odd and both } x_3 \text{ and } x_4 \text{ even}]=12\alpha'(m)$. Thus

$$\alpha'(m) = \phi(2m)/20. \tag{10}$$

Now $f_1=m$ implies that one of x_1, x_2, x_3, x_4 is incongruent to the other three modulo 2, that is,

$$M_1(m) = 4N[m=f_1;\ x_1\not\equiv x_2\equiv x_3\equiv x_4 \pmod 2] = 4\alpha'(m)$$

and

$$M_1(m) = \phi(2m)/5. \tag{11}$$

The equation $f_1'=m$ implies that just three of x_1, x_2, x_3, x_4 are odd, or just one is odd, according as $m\equiv 3$ or $1 \pmod 4$. Thus, using (9.3),

$$M_2'(m) = b\phi(m), \quad \text{where } b = 1/10, 3/5, 3/7, \tag{12}$$

according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

We note that

$$M_4(m) = N[m=f_0;\ x_1 \text{ odd},\ x_2\equiv x_3,\ x_4\equiv x_5 \pmod 2].$$

It is therefore true that $M_4(m)=2M_2'(m)$, $M_2'(m)/3$ or

* For complete results see case II below.

$M_2'(m)/3+\phi'(m)$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$. Thus, from (12) and (4),

$$M_4(m) = c\phi(m), \tag{13}$$

where $c=3/7$ or $1/5$ according as $m\equiv 5 \pmod 8$ or $\not\equiv 5 \pmod 8$.

Also $m=f_2'$ implies that all of x_1, x_2, x_3 are odd, or just one is odd according as $m\equiv 3$ or $1 \pmod 4$ and thus $M_{22}(m)=M_2'(m)$ or $M_2'(m)/3$ respectively. We have

$$M_{22}(m) = a\phi(m), \tag{14}$$

where a is defined in (6). Since $f_2=m$ implies that just one of x_1, x_2, x_3 is odd, or all are odd, we see that

$$M_2(m) = N'(m) + 3M_{22}(m),$$

where $N'(m)=N[m=f_2;\ x_1x_2x_3 \text{ odd}]$. Now

$$N'(m) = M_2'(m),\ 0,\ \text{or}\ \phi'(m),$$

according as $m\equiv 3 \pmod 4$, $\equiv 1$, or $5 \pmod 8$. Using (12), (14) and (4), we have

$$M_2(m) = d\phi(m), \tag{15}$$

where $d=2/5$, $3/5$, or $5/7$, according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$.

The following results are obvious:

$$M_{31}(m) = \alpha'(m) = \phi(2m)/20; \tag{16}$$

$$M_3(m) = 2M_{31}(m) = \phi(2m)/10; \tag{17}$$

$$M_3'(m) = 2M_2'(m)/3 = 2\phi(m)/5 \text{ or } 2\phi(m)/7, \tag{18.1}$$

according as $m\equiv 1$ or $5 \pmod 8$;

$$M_3'(4n+3) = 0; \tag{18.2}$$

$$M_4'(m) = \tfrac{1}{2}M_3'(m) = e\phi(m), \tag{19}$$

where $e=0$, $1/5$, or $1/7$ according as $m\equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

$$M_{21}(m) = 2M_{22}(m) = 2a\phi(m), \tag{20}$$

where a is defined in (6).

CASE II: n even. We express $N[n=f]$ in terms of λ and ϕ from which, by reference to formulas (9) and (8), $N[n=f]$ may

be expressed in terms of ϕ alone. Since $f_1' = 4n$ implies $x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod 2$,

(21) $\qquad M_1(2n) = \lambda(4n), \quad$ (for $M_1(m)$ see (11)),

(22) $\qquad M_3(2n) = \phi(n), \quad$ (for $M_3(m)$ see (17)).

Since

$$\phi(2n) = N[4n = f_3; x_1 \equiv x_2 \pmod 2] = M_2(2n) + 2N'(2n),$$

where $N'(2n) = N[2n = f_1'; x_1 \text{ odd}]$, we see that

$$N'(2m) = 3N[2m = f_3'] = 6M_{31}(m),$$

and that

$$N'(4n) = \lambda'(4n).$$

Using (16) and (1), we then have

(23.1) $\quad M_2(2m) = 2\phi(2m)/5$, (for $M_2(m)$ see (15)),

(23.2) $\quad M_2(4n) = \phi(4n) - 2\lambda'(4n) = \{3\phi(4n) + 2\phi(n)\}/5.$

Obviously,

(24) $\qquad M_4(2n) = M_1(n) = \phi(2n)/5$ or $\lambda(2n)$,

according as n is odd or even. (For $M_4(m)$ see (13).)

Now $M_2'(2m) = 3N[2m = f_3'] = 6M_{31}(m)$. Thus, using (16), we get

(25.1) $\qquad M_2'(2m) = 3\phi(2m)/10.$

For $M_2'(m)$, see (12).

(25.2) $\qquad M_2'(4n) = \phi(n).$

Also $M_1'(2n) = M_3(n)$, using (17),

(26) $\qquad M_3'(2n) = \phi(2n)/10$ or $\phi(n/2)$,

according as n is odd or even. (For $M_3'(m)$, see (18).)

The following results are obvious:

(27) $\qquad M_4'(2n) = 0$ or $\phi(n/2)$,

according as n is odd or even. (For $M_4'(m)$ see (19).)

(28) $\qquad M_{21}(2n) = M_1(n) = \phi(2n)/5$ or $\lambda(2n)$,

according as n is odd or even. (For $M_{21}(m)$, see (20).)

$$M_{22}(2n) = M_3(n) = \phi(2n)/10 \text{ or } \phi(n/2), \tag{29}$$

according as n is odd or even. (For $M_{22}(m)$, see (14).)

$$M_{31}(2m) = M_2(m) = d\phi(m), \tag{30.1}$$

where d is defined in (15).

$$M_{31}(4n) = 2\phi(2n)/5 \text{ or } \{3\phi(2n) + 2\phi(n/2)\}/5, \tag{30.2}$$

according as n is odd or even. (For $M_{31}(m)$, see (16).)

7. *A Reduction Formula for $\alpha'(n)$.* A reduction formula for $\alpha'(n)$ will later be found nceessary. We see that

$$\alpha'(2m) = 2N[2m = \mu_1^2 + \mu_2^2 + 8x_3^2 + 4x_4^2 + 4x_5^2] = 4M_{22}(m)$$

and

$$\begin{aligned}\alpha'(4n) &= 2N[2n = \mu_1^2 + \mu_2^2 + 2\mu_3^2 + 4x_4^2 + 4x_5^2] \\ &= 4N[2n = f_{21};\ x_1x_2 \text{ odd}] = 8\alpha'(n).\end{aligned}$$

Thus, using (10), we have

$$\alpha'(4^\alpha m) = 8^\alpha\phi(2m)/20,\ \alpha \geqq 0, \tag{31.1}$$

$$\alpha'(4^\alpha \cdot 2m) = 4 \cdot 8^\alpha a\phi(m), \tag{31.2}$$

where a is defined in (6) and $\alpha \geqq 0$.

8. *M_{11}, M_{13}, M_{12} Expressed in Terms of α and ϕ.** It is clear that $M_{11}(2n) = 6N'(n) + M_4(n)$ where $N'(n) = N[n = f_{21};\ x_1 \text{ odd}]$ $= M_{22}(n)$ or $2\alpha'(n/2)$ according as n is odd or even, and

$$M_{11}(m) = \alpha(m), \quad \text{by definition,} \tag{32.1}$$

$$M_{11}(2m) = g\phi(m), \quad \text{where } g = 4/5,\ 7/5, \text{ or } 9/7, \tag{32.2}$$

according as $m \equiv 3 \pmod 4$, $\equiv 1 \pmod 8$, or $\equiv 5 \pmod 8$;

$$M_{11}(4n) = 12\alpha'(n) + M_4(2n). \tag{32.3}$$

Now

$$\begin{aligned}\alpha(m) &= N[m = f_{11};\ x_2 \equiv x_3 \pmod 2] \\ &\quad + 2N[m = f_{12};\ x_2 \equiv 1 \pmod 2] = \alpha'(m) + 2M_{13}(m),\end{aligned}$$

* In many cases, to save space, results are expressed in terms of M_4, M_4', M_{21}, M_{22}, M_{31}, α' which have been previously expressed in terms of ϕ.

(33.1) $M_{13}(m) = \frac{1}{2}\{\alpha(m) - \alpha'(m)\} = \alpha(m)/2 - \phi(2m)/40,$

(33.2) $M_{13}(2n) = M_4(n),$

(34.1) $M_{12}(m) = 2M_{13}(m) = \alpha(m) - \phi(2m)/20,$

(34.2) $M_{12}(2n) = M_2(n).$

9. *M_{abc} Expressed in Terms of α, β' and ϕ.**

If $n = m \equiv 1 \pmod 4$,

$$M_{13}(m) = M_{444}(m) = M_{122}(m)/2.$$

If $n = m \equiv 3 \pmod 4$,

$$\begin{aligned} M_1(m) &= 4N[m = f_{11};\ x_1x_2x_3 \text{ odd}] + 4M_{13}(m) \\ &= 8N[m = f_{224};\ x_1x_2 \text{ odd}] + 4M_{13}(m) \\ &= 8M_{248}(m) + 4M_{13}(m) = 2M_{122}(m) + 4M_{13}(m). \end{aligned}$$

(35.1) $M_{122}(m) = \alpha(m) - \phi(2m)/20$ if $m \equiv 1 \pmod 4$,

(35.2) $M_{122}(m) = 3\phi(2m)/20 - \alpha(m)$ if $m \equiv 3 \pmod 4$,

(35.3) $M_{122}(2n) = \lambda(n).$

Noting that $M_{111}(m) = 4M_{224}(m) = 2M_{122}(m)$, we find

(36.1) $$M_{111}(m) = 2\alpha(m) - \phi(2m)/10,$$

if $m \equiv 1 \pmod 4$;

(36.2) $$M_{111}(m) = 3\phi(2m)/10 - 2\alpha(m),$$

if $m \equiv 3 \pmod 4$;

(36.3) $$M_{111}(2m) = 6M_{21}(m) = 12a\phi(m),$$

where a is defined in (6);

(36.4) $$M_{111}(4n) = \lambda(2n).$$

If $m \equiv 5 \pmod 8$,

$$M_{448}(m) = N[m = f_4';\ x_4 \equiv x_5 \pmod 2]$$

and

$$\begin{aligned} &4N[m = f_4';\ x_2 \not\equiv x_3 \equiv x_4 \equiv x_5 \pmod 2] \\ &\quad = 4N[m = x_1^2 + 4\mu_2^2 + 16x_3^2 + 4x_4^2 + 4x_5^2] = 2M_{244}(m). \end{aligned}$$

* See note on §8.

Then

(37.1) $M_{244}(m) = \beta(m)$,

if $m \equiv 1 \pmod 8$ by definition;

(37.2) $M_{244}(m) = M_4'(m)/2 = \phi(m)/14$.

if $m \equiv 5 \pmod 8$;

(37.3) $M_{244}(m) = M_{22}(m)/2 = \phi(m)/20$

if $m \equiv 3 \pmod 4$;

(37.4) $M_{244}(2n) = M_{31}(n)$.

It is clear that

$$M_{112}(m) = 3M_{244}(m) + N'(m),$$

where $N'(m) = N[m = f_{112};\ x_1x_2x_3 \text{ odd}]$.

Now

$$N'(m) = 2N[m = \mu_1^2 + 2\mu_2^2 + 8x_3^2 + 2x_4^2 + 8x_5^2] = M_{22}(m),$$

$\phi'(m)/2$, or 0, according as $m \equiv 3$, 5, or $\pm 1 \pmod 8$. Moreover,

$$M_{112}(2n) = M_{31}(n) + 3N[2n = f_{124};\ x_1x_2 \text{ odd}] = M_{31}(n) + 6N''(n)$$

where $N''(n) = N[n = f_{12};\ x_1 \text{ odd}]$. Now $N''(n) = M_{13}(n)$ if n is odd, $2M_{22}(n/2)$ if $n \equiv 2 \pmod 4$, or $4\alpha'(n/4)$ if $n \equiv 0 \pmod 4$.

(38.1) $M_{112}(m) = 3\beta(m),\ 5\phi(m)/14,\ \phi(m)/4,$ or $3\phi(m)/20,$

according as $m \equiv 1$, 5, 3, or 7 (mod 8);

(38.2) $M_{112}(2m) = 3\alpha(m) - \phi(2m)/10$;

(38.3) $M_{112}(4m) = (d + 12a)\phi(m)$,

where d and a are defined in (15) and (6), respectively;

(38.4) $M_{112}(8n) = M_{31}(4n) + 24\alpha'(n)$;

(39.1) $M_{114}(m) = 3M_{111}(m)/4 = 3\alpha(m)/2 - 3\phi(2m)/40$ if $m \equiv 1 \pmod 4$;

(39.2) $M_{114}(m) = M_{111}(m)/4 = 3\phi(2m)/40 - \alpha(m)/2$ if $m \equiv 3 \pmod 4$;

(39.3) $M_{114}(2m) = 3M_{21}(m) = 6a\phi(m)$,

where a is defined in (6);

(39.4) $\qquad M_{114}(4n) = M_1(n).$

Noting that $M_{118}(m) = 2N[m = f_{288};\ x_2 \text{ odd}] + 3M_{448}(m)$, we have

(40.1) $\qquad M_{118}(m) = 2M_{244}(m) = \phi(m)/10,$

if $m \equiv 3 \pmod 8$;

(40.2) $\qquad M_{118}(m) = 3M_{244}(m) = 3\beta(m) \text{ or } 3\phi(m)/14,$

according as $m \equiv 1$ or $5 \pmod 8$;

(40.3) $\qquad M_{118}(m) = 0 \text{ if } m \equiv 7 \pmod 8;$

(40.4) $\qquad M_{118}(2m) = 3M_{12}(m) = 3\alpha(m) - 3\phi(2m)/20;$

(40.5) $\qquad M_{118}(4n) = M_2(n).$

Noting that $M_{124}(m) = 2M_{244}(m)$, we have,

(41.1) $\qquad M_{124}(m) = 2\beta(m),\ \phi(m)/7, \text{ or } \phi(m)/10,$

according as $m \equiv 1 \pmod 8$, $\equiv 5 \pmod 8$, or $\equiv 3 \pmod 4$;

(41.2) $\qquad M_{124}(2m) = \alpha(m);$

(41.3) $\qquad M_{124}(4m) = g\phi(m),$

where g is defined in (32.2);

(41.4) $\qquad M_{124}(8n) = 12\alpha'(n) + M_4(2n).$

Now $M_{128}(m) = M_{122}(m)/2$ if $m \equiv 3 \pmod 4$. But if $m \equiv 1 \pmod 4$,

$$\begin{aligned} M_1(m) &= N[m = f_1;\ \text{just three of } x_1, x_2, x_3, x_4 \text{ odd}] \\ &\quad + N[m = f_1;\ \text{just one of } x_1, x_2, x_3, x_4 \text{ odd}] \\ &= 8N[m = f_{224};\ x_1x_2x_3 \text{ odd}] + 4M_{13}(m). \end{aligned}$$

Also

$$M_{122}(m) = M_{188}(m) + 2N[m = f_{224};\ x_1x_2x_3 \text{ odd}].$$

Therefore

$$M_1(m) - 4M_{122}(m) = 4M_{13}(m) - 4M_{128}(m);$$

and

(42.1) $\qquad M_{128}(m) = 3\phi(2m)/40 - \alpha(m)/2,$

if $m \equiv 3 \pmod 4$;

(42.2) $M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8$,

if $m \equiv 1 \pmod 4$;

(42.3) $M_{128}(2n) = M_2'(n)$.

Since $M_{144}(m) = M_{12}(m)$, if $m \equiv 1 \pmod 4$, we have

(43.1) $M_{144}(m) = \alpha(m) - \phi(2m)/20$ or 0,

according as $m \equiv 1$ or $3 \pmod 4$;

(43.2) $M_{144}(2n) = M_{21}(n)$;

(44.1) $M_{148}(m) = 2M_{244}(m) = 2\beta(m)$ or $\phi(m)/7$,

according as $m \equiv 1$ or $5 \pmod 8$;

(44.2) $M_{148}(m) = 0$ if $m \equiv 3 \pmod 4$;

(44.3) $M_{148}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20$;

(44.4) $M_{148}(4n) = M_2(n)$;

(45.1) $M_{188}(m) = M_{128}(m) = 3\alpha(m)/2 - \phi(2m)/8$,

if $m \equiv 1 \pmod 4$;

(45.2) $M_{188}(m) = 0$ if $m \equiv 3 \pmod 4$;

(45.3) $M_{188}(2n) = M_2'(n)$;

(46.1) $M_{222}(m) = M_{112}(m)/3$, $3M_4(m)/4$, $M_4(m)/2$, or $M_4(m)/4$,

that is, $\beta(m)$, $3\phi(m)/20$, $3\phi(m)/14$ or $\phi(m)/20$, according as $m \equiv 1$, 3, 5, or $7 \pmod 8$;

(46.2) $M_{222}(2m) = \alpha(m)$;

(46.3) $M_{222}(4m) = g\phi(m)$,

where g is defined in (32.2);

(46.4) $M_{222}(8n) = 12\alpha'(n) + M_4(2n)$;

(47.1) $M_{224}(m) = M_{122}(m)/2 = \alpha(m)/2 - \phi(2m)/40$.

if $m \equiv 1 \pmod 4$;

(47.2) $M_{224}(m) = 3\phi(2m)/40 - \alpha(m)/2$.

if $m \equiv 3 \pmod 4$;

(47.3) $M_{224}(2n) = M_{21}(n)$.

Since $M_{228}(m) = M_{244}(m)$ or $2M_{244}(m)$, according as $m \equiv 1 \pmod 4$ or $3 \pmod 8$,

(48.1) $M_{228}(m) = \beta(m)$ or $\phi(m)/14$,

according as $m \equiv 1$ or $5 \pmod 8$;

(48.2) $M_{228}(m) = \phi(m)/10$ or 0,

according as $m \equiv 3$ or $7 \pmod 8$;

(48.3) $M_{228}(2m) = M_{12}(m) = \alpha(m) - \phi(2m)/20$;

(48.4) $M_{228}(4n) = M_2(n)$;

(49.1) $M_{248}(m) = M_{128}(m)/2 = 3\alpha(m)/4 - \phi(2m)/16$,

if $m \equiv 1 \pmod 4$;

(49.2) $M_{248}(m) = 3\phi(2m)/80 - \alpha(m)/4$,

if $m \equiv 3 \pmod 4$;

(49.3) $M_{248}(2n) = M_{22}(n)$.

Since $M_{288}(m) = M_{222}(m)/3$ or $M_{222}(m)$, according as $m \equiv 3$ or $1 \pmod 8$;

(50.1) $M_{288}(m) = \beta(m)$, $\phi(m)/20$, 0, or 0,

according as $m \equiv 1$, 3, 5, or $7 \pmod 8$;

(50.2) $M_{288}(2m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40$;

(50.3) $M_{288}(4n) = M_4(n)$;

(51.1) $M_{444}(m) = M_{13}(m) = \alpha(m)/2 - \phi(2m)/40$,

if $m \equiv 1 \pmod 4$;

(51.2) $M_{444}(n) = 0$ if $n \equiv 2$ or $3 \pmod 4$;

(51.3) $M_{444}(4n) = M_1(n)$.

Since $M_{448}(m) = M_{148}(m)/2$ if $m \equiv 1 \pmod 4$, we have

(52.1) $M_{448}(m) = \beta(m)$ or $\phi(m)/14$,

according as $m \equiv 1$ or $5 \pmod 8$;

(52.2) $M_{448}(n) = 0$ if $n \equiv 2$ or $3 \pmod 4$;

(52.3) $M_{448}(4n) = M_2(n)$;

(53.1) $M_{488}(m) = M_{248}(m) = 3\alpha(m)/4 - \phi(2m)/16$,

if $m \equiv 1 \pmod 4$;

(53.2) $M_{488}(n) = 0$ if $n \equiv 2$ or $3 \pmod 4$;

(53.3) $M_{488}(4n) = M_3(n)$;

(54.1) $M_{888}(m) = M_{288}(m) = \beta(m)$,

if $m \equiv 1 \pmod 8$;

(54.2) $M_{888}(n) = 0$,

if $n \equiv 2$ or $3 \pmod 4$ or $5 \pmod 8$;

(54.3) $M_{888}(4n) = M_4(n)$.

10. *Some Miscellaneous Results.* Letting f'_{abc} denote the form

$$x_1^2 + ax_2^2 + bx_3^2 + cx_4^2 + 16x_5^2$$

and

$$M'_{abc}(n) = N\left[f'_{abc} = n\right],$$

we have

(55) $M'_{114}(8n + 7) = M'_2(8n + 7)/2 = \phi(8n + 7)/20$;

(56.1) $M'_{111}(8n + 7) = 4M'_{114}(8n + 7) = \phi(8n + 7)/5$;

(56.2) $M'_{111}(4m) = 8M_{22}(m) + \lambda(m) = 12a\phi(m)$,

where a is defined in (6);

(56.3) $M'_{111}(8n) = \lambda(2n)$;

(57.1) $M'_{248}(8n + 7) = M_{244}(8n + 7)/2 = \phi(8n + 7)/40$;

(57.2) $M'_{248}(8n + 5) = M_{448}(8n + 5)/2 = \phi(8n + 5)/28$;

(58) $M'_{488}(8n + 5) = M'_{248}(8n + 5) = \phi(8n + 5)/28$.

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A SERIES OF RATIONAL SURFACES

BY A. R. WILLIAMS

Montesano* has considered a rational sextic surface $\sum_6$, whose plane system is the web of nonics having 8 fixed triple points and 3 simple points. The double curve is composed of a triple line and 3 double lines. The latter are coplanar and concurrent at a point on the triple line, the intersection being a quadruple point of the surface. This surface is one of a series whose orders are successive multiples of three. The double curve in each case is composed of a k-fold line, of multiplicity 3 less than the order of the surface, and k double lines which all meet the k-fold line, but which, except in the $\sum_6$ just mentioned, are skew to each other. From each surface may be derived by quadratic transformation another of degree one lower, whose double curve consists of a $(k-1)$-fold line and $k-2$ double lines, and which has an isolated triple point. The quintic so obtained from $\sum_6$ is exceptional in that it has 2 consecutive skew double lines (instead of 1), and its triple point is on the double line. I have described that surface in a previous paper.†

A curve of order $3r$, $r \geqq 3$, having 8 fixed r-fold points, $A_1, \cdots, A_8$, an $(r-3)$-fold point at A_0, the ninth base point of the pencil of cubics determined by $A_1, \cdots, A_8$, and $3(r-2)$ fixed simple points has 3 degrees of freedom. Two curves of this web have $3(r-1)$ variable intersections. Such a web defines a rational surface whose plane sections correspond to the curves of the web. For $r=2$ we have indeed the web of sextics with 8 fixed double points. But this case is altogether exceptional, since any two curves of this web which pass through an arbitrary point pass also through another point determined by the first. The corresponding surface is in fact a quadric cone. It will be convenient to consider independently the case $r=4$. There are ∞^4 curves of order 12 that have in common 8 fixed quadruple points and 6 fixed simple points. There

* Montesano, Rendiconti di Napoli, vol. 46 (1907), p. 66.

† This Bulletin, vol. 34 (1928), p. 761.

is a single nonic c_9 having triple points at the 8 points $A_1, \cdots, A_8$ and simple points at the 6 points $B_1, \cdots, B_6$. Corresponding to this family of duodecimics we have in 4-space a surface of order 10, whose hyperplane sections correspond to the curves of the family. One pencil of hyperplane sections correspond to the duodecimics composed of the nonic c_9 and the pencil of cubics through $A_1, \cdots, A_8$. Since the cubics of this pencil have 4 variable intersections with the duodecimics of the family, this pencil of hyperplane sections have in common a sextic curve with 6 nodes lying in a plane π and a point A_0' in π, but not the sextic, which corresponds to A_0, the ninth base point of the pencil of cubics. Similarly there is a pencil of hyperplane sections corresponding to the duodecimics made up of the cubic c_3^i, determined by B_i, and the pencil of nonics n_i, which have the 8 points A for triple points and pass through the points B, except B_i. To the cubic c_3^i corresponds on the surface a cubic lying in a plane σ_i. The hyperplane section corresponding to c_3^i and c_9 consists of the sextic in π, the cubic in σ_i, and the line corresponding to B_i. This hyperplane section belongs to both pencils and hence π and σ_i have a line s_i in common which passes through A_0'. This line meets the plane sextic in π in two points corresponding to the points in which c_3^i meets c_9, and in 4 more points which correspond to the 4 remaining base points of the pencil of nonics n_i. To a nonic of this pencil corresponds a skew septimic which lies in a hyperplane containing σ_i, and meets σ_i in the 4-points where s_i meets the sextic in π, and in 3 variable points (not in π) corresponding to the intersections of the nonic with the cubic c_3^i. To a general cubic through the 8 points A corresponds a skew quartic which lies in a hyperplane containing π, and which meets π in A_0' and 3 variable points that correspond to the intersections of the cubic with c_9. We have thus in 4 space a plane π containing a sextic belonging to the surface, and 6 planes σ each containing a cubic belonging to the surface. The 6 planes σ meet π in 6 lines which have in common A_0'. The cubic in each plane σ passes through A_0'.

If now we project the surface from A_0' upon a hyperplane S_3 we obtain a surface $\sum_9$, of order 9, having a 6-fold line k, the trace of π in S_3, and 6 double lines, $d_1, \cdots, d_6$, all of which meet the 6-fold line, but are skew. There are 6 simple lines, the

projections of the lines corresponding to the points B, each lying in a plane with the 6-fold line and one of the double lines. The tangent plane at A_0' gives another line l_0 on the surface in S_3, that meets the 6 double lines, but not the 6-fold line. This configuration in 4-space seemed worthy of notice. The result of the projection might of course have been obtained directly by use of the web of duodecimics having quadruple points at the 8 points A, simple points at the 6 points B, and a simple point at A_0. To the 6-fold line corresponds the nonic c_9 having triple points at $A_1, \cdots, A_8$ and simple points at $B_1, \cdots, B_6$. To a double line corresponds a cubic of the pencil determined by a point B. The genus of a plane section of $\sum_9$ is 7, and is accounted for by the 6-fold line and the 6 double lines. The class of the surface is $3(12-1)^2$ less 39 for each quadruple point, that is, 51. The curve of contact of the tangent cone meets each double line in 3 variable points and 4 fixed points which are pinch points. It meets the 6-fold line in 3 variable points and 18 pinch points. The order of the tangent cone from an arbitrary point not on the surface is 30. Its genus, which is that of the Jacobian of the corresponding net of basic curves, is 49. Hence it has 105 stationary edges and 252 double edges.

If we now apply a quadratic transformation whose fundamental system is a general point O on one of the double lines, say d_5, and the degenerate conic consisting of the 6-fold line k and any other double line, say d_6, the resulting surface is $\sum_8$, of order 8. For it loses the plane kO, or kd_5, six times, the plane d_6O twice, and the plane kd_6 twice. The transform of a general plane of the pencil k is a plane of the pencil d_6, and the relation between these planes is a collineation. Therefore the planes through d_6 meet the new surface in cubics, and d_6 is a 5-fold line on $\sum_8$. The line k is a simple line on $\sum_8$ corresponding to l_5, the residual intersection of the plane kd_5 with $\sum_9$. A plane through O is invariant as a whole. Its points are transformed quadratically. A line meeting d_6 and d_5 meets $\sum_9$ five more times. Since it lies in a plane through O and in a plane of the pencil d_6, its transform is a line passing through the intersection on k and d_5 and meeting $\sum_8$ in 5 more points. Hence the point kd_5 is a triple point of $\sum_8$. The section of $\sum_8$ by the plane kd_6 consists of the 5-fold line d_6, the simple line k, and 2 simple lines e_1 and e_2, which pass

through the triple point and correspond to the tangent planes to $\sum_9$ at O. The line l_0 on $\sum_9$ meets d_6 and d_5, and its transform is therefore a line l_0' through the triple point. A general quadric surface intersects $\sum_9$ in a curve whose image is of order 24 and has 8-fold points at $A_1, \cdots, A_8$, and double points at the 6 points B and at A_0. Hence the plane system of $\sum_8$ is the web of duodecimics that have quadruple points at $A_1, \cdots, A_8$, simple points at $B_1, \cdots, B_5$ and at A_0, and pass also through E_1 and E_2, the two points on c_3' (the cubic through B_5) that correspond to O. The plane d_6O meets $\sum_9$ in a septimic having a double point at O. The image of that section is a nonic having triple points at $A_1, \cdots, A_8$ and passing through the 5 points $B_1, \cdots, B_5$, and the points E_1 and E_2. Hence this same nonic is the image of the 5-fold line of $\sum_8$. To a plane section through the 5-fold line corresponds a cubic of the pencil $A_1, \cdots, A_8$. Twelve of these have a node. The image of the triple point of $\sum_8$ is the cubic c_3^5 containing the points B_5, E_1, E_2, and A_0. To these points correspond the nodal lines k, e_1, e_2, and l_0'. The first 3 lie in the plane kd_6. To the remaining double lines of $\sum_9$ correspond 4 double lines of $\sum_8$, d_1', d_2', d_3', d_4', meeting the 5-fold line. Their images are as before the cubics c_3^1, c_3^2, c_3^3, c_3^4 determined by $B_1, \cdots, B_4$. To plane sections of $\sum_8$ through the triple point correspond the net of nonics having triple points at $A_1, \cdots, A_8$ and simple points at $B_1, \cdots, B_4$. To the pencil of plane sections through a double line, say d_1', correspond the pencil of nonics having triple points at $A_1, \cdots, A_8$, and passing through $B_2, B_3, B_4, B_5, E_1, E_2$. To the section containing d_1' and the node corresponds the sextic having double points at $A_1, \cdots, A_8$ and passing through $B_2B_3B_4$. This section contains l_0'. There is thus a difference between $\sum_8$ and $\sum_9$ in the number of plane sections through a double line that have a node. In both cases to such a pencil of plane sections correspond a pencil of nonics having 8 fixed triple points. But for $\sum_9$ the remaining base points of the pencil are of general position, and the number of curves of the pencil that have an extra node is 32. For $\sum_8$ three base points are on a cubic through the 8 triple points. One nonic of the pencil is composed of this cubic and a sextic. It is in fact the image of the section through the double line and the triple point. The two points, apart from the base points, where the sextic meets

the cubic are to be deducted, and the number is 30. It is easy to see that in the next case, $r=5$, this difference is 3, and that it does not increase further with r. For the degenerate curve consists of a cubic, image of the triple point, and a curve of order $3r-6$, which has $(r-2)$-fold points at $A_1, \cdots, A_8$ and an $(r-5)$-fold point at A_0, and hence meets the cubic in 3 other points. The class of $\sum_8$ is 48, or 3 less than the class of $\sum_9$. The Jacobian of a net contains the factor c_3^5, the image of the node. Removing it, we find that the curve of contact of the tangent cone from an arbitrary point not on the surface, meets each of the 4 double lines in 3 variable points and 4 pinch points, and meets the 5-fold line in 3 variable points and 16 pinch points, the last number being 2 less than for the 6-fold line of $\sum_9$. The Jacobian meets the image of the node in 6 variable points; and hence the tangent cone has a 6-fold edge. The order of the tangent cone, its genus, and the number of its stationary edges are less by 2, 7, and 15 respectively than in $\sum_9$. These differences are all constant in passing from $\sum_{3r-3}$ to $\sum_{3r-4}$. The difference in the number of double edges is a function of r.

The stationary tangent planes to the cubic cone at the triple point of $\sum_8$ are of some interest. The sections of the surface by these planes correspond to nonics of the above mentioned net that have 3-point contact with the cubic c_3^5. Such a point of contact is a possible 9th triple point for nonics having triple points at $A_1, \cdots, A_8$. A curve of order $3r$ having 8 fixed r-fold points can not have a 9th r-fold point assigned arbitrarily. Halphen* has shown that for $r=3$ the locus of the 9th triple point is a curve that meets any cubic of the pencil in 8 points aside from the base points. A nonic having triple points at $A_1, \cdots, A_8$ and touching the cubic at one of these points will have 3-point contact there with the cubic. The 9th stationary tangent plane is accounted for by A_0. A nonic having the points $A_1, \cdots, A_8$ for triple points and passing through A_0 meets a cubic of the pencil in 2 more points, either of which corresponds to the other in the involution I_{17} determined by the sextics that have double points at $A_1, \cdots, A_8$. Hence such a nonic tangent to a cubic of the pencil at A_0 must have 3-point

* Bulletin de la Société Mathématique de France, vol. 10 (1881), p. 162; or *Oeuvres*, vol. 2, p. 547.

contact there. This insures the reality of one stationary tangent plane.*

We have seen that in either $\sum_8$ or $\sum_9$ a plane through a double line, or through the 6-fold or the 5-fold line, meets the surface in a curve having 3 variable intersections with the multiple line. When 2 such points coincide the plane of the section is a stationary plane in the developable of the stationary tangent planes of the surface; and the point is a point of contact of the parabolic curve with the multiple line. For example, the images of the sections of $\sum_9$ by planes through the 6-fold line are the cubics of the pencil $A_1, \cdots, A_8$. Let ϕ and ψ be two of them. To find how many cubics of the pencil are tangent to the image of the 6-fold line, that is, the nonic c_9 having triple points at $A_1, \cdots, A_8$ and passing through the 6 points B, we have merely to find the number of intersections of the Jacobian of ϕ, ψ, and c_9 not accounted for at the A's. This Jacobian is of order 12 and has a triple point at each of the points $A_1, \cdots, A_8$ whose 3 tangents coincide with those of c_9.† Hence there are 12 cubics of the pencil tangent to c_9, and therefore 12 such points on the 6-fold line. Similarly there are 6 such points on each double line. Also on $\sum_8$ there are 12 such points on the 5-fold line and 6 on each double line. The genus of a curve of order n on $\sum_9$ has the upper limit $(4n^2+12n+135)/72$. On $\sum_8$ this limit is $(n^2+4n+34)/16$.

The system of rational quartics on the two surfaces deserves notice. To the sextics of the web having double points at $A_1, \cdots, A_8$ correspond octavic curves. Such a sextic may be composed of the line joining 2 of the points A and the quintic having double points at the other 6 and passing through the first 2. The line and the quintic intersect in 3 more points invariant under I_{17}. We get thus on either surface 2 rational quartics intersecting in 3 points, and on $\sum_8$ passing each once through the node. There are 28 such pairs. Similarly the sextic may be composed of the conic through 5 of the points A and the quartic having double points at the other 3 and passing through the first 5. This gives 56 pairs of similar quartics. The quartics corresponding to the points $A_1, \cdots, A_8$ are of this type. With the quartic corresponding to A_i is paired the

* See Snyder, American Journal of Mathematics, vol. 33 (1911), p. 328.
† Hilton, *Higher Plane Curves*, p. 110.

quartic (also rational) whose image is the sextic having a triple point at A_i and double points at the other 7. This sextic corresponds to A_i in I_{17}. There are thus 92 pairs of such quartics on both surfaces.

The extension of the above is obvious. For the sake of completeness a summary of the general case is added. The following applies to $r \geqq 4$. There is a web of curves of order $3r$, and of genus $3r-5$, which have in common 8 r-fold points $A_1, \cdots, A_8$, $3(r-2)$ simple points B, and an $(r-3)$-fold point at A_0. Two curves of the web have $3(r-1)$ variable intersections. There is one curve c_{3r-3} which has $(r-1)$-fold points at $A_1, \cdots, A_8$, passes through the $3(r-2)$ points B, and has an $(r-4)$-fold point at A_0. We have, therefore, a rational surface $\sum_{3r-3}$, whose plane sections correspond to the curves of the web. The double curve of this surface consists of a $(3r-6)$-fold line and $3r-6$ double lines, which meet the former, but are skew to each other. The surface has also $3r-6$ simple lines and a rational curve of order $r-3$, which meets each double line once and the $(3r-6)$-fold line $r-4$ times. The image of the $(3r-6)$-fold line is c_{3r-3}. The images of the double lines are the cubics of the pencil $A_1, \cdots, A_8$ determined respectively by the points B. The class of $\sum_{3r-3}$ is $3(6r-7)$. The curve of contact of the tangent cone from an arbitrary point not on the surface, meets each double line in 3 variable points and 4 pinch points. It meets the $(3r-6)$-fold line in 3 variable points and $6(2r-5)$ pinch points. The order of the tangent cone is $6(2r-3)$. Its genus is $24r-47$; and it has $3(18r-37)$ stationary edges, and $12(6r^2-26r+29)$ double edges.

Applying a quadratic transformation whose fixed conic is the $(3r-6)$-fold line and a double line d_i, and whose fixed point O is on another double line d_j, we obtain a surface $\sum_{3r-4}$ of order one lower, which has a triple point at the intersection of d_j with the $(3r-6)$-fold line, and whose double curve is a $(3r-7)$-fold line, coinciding with d_i, and $3r-8$ double lines meeting the former, but skew to each other. There are $3r-8$ simple lines and a rational curve of order $r-3$ which passes once through the triple point and meets each double line once and the $(3r-7)$-fold line $r-4$ times. The class of the new surface is less by 3. As in $\sum_{3r-3}$, the curve of contact of the tangent cone of $\sum_{3r-4}$ has 3 variable intersections with each double line

and with the $(3r-7)$-fold line, and there are, as before, 4 pinch points on each double line. But the number of pinch points on the $(3r-7)$-fold line is 2 less. The curve of contact of the tangent cone passes 6 times through the triple point. As remarked above, the order of the tangent cone, its genus, and the number of its stationary edges are less by 2, 7, and 15 respectively than in $\sum_{3r-3}$. The number of double edges in $\sum_{3r-4}$ is $3(24r^2-112r+137)$, a reduction of $24r-63$. The plane system of $\sum_{3r-4}$ is the same as that of $\sum_{3r-3}$ with the exception that B_i is dropped, and two associated points on the cubic which is the image of d_j are added. The image of the $(3r-7)$-fold line is the curve c'_{3r-3} which has $(r-1)$-fold points at $A_1, \cdots, A_8$, passes through the B's (except B_i) and through the 2 associated points just mentioned, and has an $(r-4)$-fold point at A_0. There are $6(r-2)$ plane sections of either surface containing the $(3r-6)$-fold or the $(3r-7)$-fold line and tangent to it. Through a double line on either surface are 6 sections that are tangent to it. The planes of these sections are doubly stationary, that is, stationary planes in the developable of the stationary tangent planes of the surface. The genus of a curve of order n on $\sum_{3r-3}$ does not exceed the greatest integer in

$$\frac{4n^2+12n(r-3)+3\ (6r^2-19r+25)}{24(r-1)}$$

The corresponding expression for $\sum_{3r-4}$ is

$$\frac{4n^2+4n(3r-8)+18r^2-51r+52}{8(3r-4)}$$

Exactly as in $\sum_9$ and $\sum_8$ there are on $\sum_{3r-3}$ and $\sum_{3r-4}$ 92 pairs of rational curves of order r. The two curves of such a pair meet 3 times on the curve whose image is the locus of invariant points in I_{17}, and in $\sum_{3r-4}$ both pass once through the triple point. To the locus of invariant points in I_{17} corresponds on either surface a curve of order $3r$, which in $\sum_{3r-4}$ passes 3 times through the triple point.

THE UNIVERSITY OF CALIFORNIA

ON INCOMPLETE NUMERICAL FUNCTIONS

BY J. V. USPENSKY

1. *Introduction.* Numerical functions depending on divisors of an integer find important applications in various questions of higher arithmetic. In general they are defined by sums of the type

$$\sum f(d, \delta)$$

extended over conjugate divisors of a given integer, these being subject, sometimes, to certain limitations. If these limitations are expressed in the form of an inequality such as $d<\delta$ or $d<3\delta$ we call the numerical function *incomplete.*

The first example of the use of such functions in an important arithmetical problem is due to Liouville. For an odd integer m let $g(m)$ be defined as a sum

$$g(m) = \sum(-1)^{(d-1)/2}$$

extended over divisors of m which are less than their conjugate; in other words, the sum extends over all representations of m in the form $m=d\delta$ where d and δ are positive integers and $d<\delta$.

If, on the other hand,

$$N(n = x^2 + y^2 + z^2)$$

denotes, as usual, the number of representations of n as a sum of three squares, then for $n\equiv 3 \pmod 8$ we have this peculiar result:

$$N(n = x^2 + y^2 + z^2) = 8[g(n) + 2g(n - 4\cdot 1^2) + 2g(n - 4\cdot 2^2) + 2g(n - 4\cdot 3^2) + \cdots].$$

In the right member, the series is continued so far as the arguments remain positive.

Aside from its obvious practical value this result of Liouville's is very curious in another respect; this may be discovered by comparing it with a different expression of the same number.

Denoting by $\rho(m)$ a complete function

$$\rho(m) = \sum(-1)^{(d-1)/2},$$

where the sum extends over all the divisors d of an odd integer m, we have also

$$N(n = x^2 + y^2 + z^2) = 8\left[\rho\left(\frac{n-1^2}{2}\right) + \rho\left(\frac{n-3^2}{2}\right) + \rho\left(\frac{n-5^2}{2}\right) + \cdots\right].$$

In both cases the right hand members are built similarly, but while in the second expression the sum in the right hand member is suggested by the fact that $4\rho(m)$ is a number of representations of m as a sum of two squares, no similar origin can be devised for the right hand member of the first expression. At least it seems that the function $g(m)$ has nothing to do with the number of representations by binary forms.

Since the time when this first result by Liouville became known several such results have been discovered, mostly by the use of elliptic functions. It is the purpose of this note to show how all similar results can be derived in the most elementary manner from certain very general formulas which I have had in my possession for years, but hitherto had no opportunity to publish in full with their implications concerning incomplete numerical functions. However, it should be noted that in my paper *On Gierster's class-number relations** important use has been made of formulas of this type.

2. *The Fundamental Formulas.* The general formulas we shall establish first contain a function $F(x, y, z)$ defined for integral values of its arguments and satisfying the following conditions of parity:

$$F(-x, y, z) = F(x, y, z),\ F(x, -y, -z) = -F(x, y, z).$$

Moreover

$$F(x, 0, z) = 0,$$

whenever the middle argument y can vanish. Aside from these requirements there are no other conditions imposed on $F(x, y, z)$.

* American Journal of Mathematics, vol. 50 (1928), No. 1.

In the selection of this function we therefore possess a high degree of freedom.

Let n be an arbitrary positive integer. We consider first the equation

$$n = i^2 + d\delta, \tag{a}$$

where i may have any integral value, while d and δ are positive integers. The sum

$$S = \sum_{(a)} F(d - 2i, \delta - d + i, i)$$

is extended over all the solutions of the equation (a). On the other hand, consider the integral solutions of the equation

$$4n = h^2 + \Delta\Delta', \tag{b}$$

where Δ and Δ' are positive integers satisfying the conditions

$$3\Delta < \Delta', \quad \Delta' + \Delta \equiv 0 \pmod 4.$$

Let us now form another sum

$$S' = \sum_{(b)} \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) - F\left(\frac{\Delta' + 3\Delta - 2h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right)\right\}.$$

There is a simple relation between the sums S and S'; namely,

$$S = S',$$

when n is not a perfect square; but if $n = s^2$,

$$S = S' + \sum_{j=1}^{s-1} \{F(2s, s - j, s) - F(2s - j, s - j, s)\}.$$

This relation can be exhibited more explicitly as follows:

$$\sum_{(a)} F(d - 2i, \delta - d + i, i)$$

$$= \sum_{(b)} \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) - F\left(\frac{\Delta' + 3\Delta - 2h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right)\right\} + T \tag{I}$$

where $T=0$, if n is not a perfect square,

$$T = \sum_{j=1}^{s-1}\{F(2s, s-j, s) - F(2s-j, s-j, s)\}, \text{ if } n = s^2.$$

This is our first general formula. The second is quite similar, but contains a function satisfying the conditions

$$F(-x, y, z) = -F(x, y, z), \qquad F(x, -y, -z) = F(x, y, z),$$
$$F(x, 0, z) = 0.$$

It can be written as follows:

$$\text{(II)} \qquad \sum_{(a)} F(d-2i, \delta-d+i, i)$$
$$= \sum_{(b)} \left\{F\left(h, \frac{\Delta'-3\Delta}{4}, \frac{\Delta-h}{2}\right) + F\left(\frac{\Delta'+3\Delta-2h}{4}, \frac{\Delta'-3\Delta}{4}, \frac{\Delta-h}{2}\right)\right\} - T,$$

where the extent of summations and the meaning of T are the same as in (I).

Two more formulas of the same kind can be added. One of them is

$$\text{(III)} \qquad \sum_{(c)} F(\delta+i, \delta+i-d, i)$$
$$= \sum_{(d)} \left\{F\left(\frac{\Delta'+\Delta}{2}, \frac{\Delta'-\Delta}{2}, \Delta-h\right) - F\left(h, \frac{\Delta'-\Delta}{2}, \Delta-h\right)\right\} + T,$$

for the function with the same conditions of parity as in (I), while for the function with the same parity as in (II) we have

$$\text{(IV)} \qquad \sum_{(c)} F(\delta+i, \delta+i-d, i)$$
$$= \sum_{(d)} \left\{F\left(\frac{\Delta'+\Delta}{2}, \frac{\Delta'-\Delta}{2}, \Delta-h\right) + F\left(h, \frac{\Delta'-\Delta}{2}, \Delta-h\right)\right\} + T.$$

In both cases the extent of the summations is as indicated below by (c) and (d):

(c) $$n = i^2 + 2d\delta,$$

(d) $$n = h^2 + \Delta\Delta', \quad \Delta < \Delta', \quad \Delta' \equiv \Delta \pmod 2,$$

and $T=0$ if n is not a perfect square, but

$$T = \sum_{j=0}^{s-1} \{F(j, j, s) - F(s, j, s)\}, \quad \text{if} \quad n = s^2.$$

3. *Proof of the Preceding Formulas.* All these formulas were first discovered by what may be termed "groping in the dark," but, once discovered, their proof did not present any difficulty, and is, indeed, extremely simple. As the proof is the same in all cases, it suffices here to develop it in one case; for instance, in the case of the formula (I).

The sum S can be split into three parts S_1, S_2, S_3 according as $\delta-d+i>0$, <0 or $=0$. The equations

$$h = d - 2i, \Delta = d, \Delta' = 4\delta - d + 4i$$

establish a one-to-one correspondence between the solutions of

(a) $$n = i^2 + d\delta$$

with $\delta-d+i>0$ and the solutions of

(b) $$4n = h^2 + \Delta\Delta'$$

satisfying the conditions

(e) $$0 < 3\Delta < \Delta', \quad \Delta' + \Delta \equiv 0 \pmod 4, \quad \Delta' - \Delta + 2h > 0.$$

The second group of equations,

$$h = \delta + 2i, \quad \Delta = \delta, \quad \Delta' = 4d - \delta - 4i,$$

establishes a one-to-one correspondence between the same solutions of (b) and the solutions of (a) with $\delta-d+i<0$. Thus

$$S_1 + S_2 = \sum_{(e)} \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) + F\left(\frac{\Delta' + 3\Delta - h}{4}, \frac{3\Delta - \Delta'}{4}, \frac{h - \Delta}{2}\right) \right\},$$

or, because $F(x, -y, -z) = -F(x, y, z)$,

$$S_1 + S_2 = \sum_{(e)} \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) - F\left(\frac{\Delta' + 3\Delta - h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right)\right\}.$$

In the right member the sum is extended over solutions of (a) satisfying the conditions (e). If, on the other hand, h, Δ, Δ' run over the solutions of (b) satisfying the conditions

(f) $\quad 0 < 3\Delta < \Delta', \Delta' + \Delta \equiv 0 \pmod 4, \Delta' - \Delta + 2h < 0,$

the numbers h_1, Δ_1, Δ_1' defined by the equations

$$h_1 = -\frac{\Delta' + 3\Delta - 2h}{4}, \Delta_1 = \frac{\Delta - \Delta' + 2h}{4}, \Delta_1' = \frac{\Delta' - \Delta - 6h}{4},$$

reproduce exactly the same set. It is obvious therefore that

$$\begin{aligned}\sigma &= \sum_{(f)} \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) - F\left(\frac{\Delta' + 3\Delta - 2h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right)\right\} \\ &= \sum_{(f)} \left\{ F\left(h_1, \frac{\Delta_1' - 3\Delta_1}{4}, \frac{\Delta_1 - h_1}{2}\right) - F\left(\frac{\Delta_1' + 3\Delta_1 - 2h_1}{4}, \frac{\Delta_1' - 3\Delta_1}{4}, \frac{\Delta_1 - h_1}{2}\right)\right\}.\end{aligned}$$

But, by the properties of $F(x, y, z)$, we have

$$\begin{aligned}F\left(h_1, \frac{\Delta_1' - 3\Delta_1}{4}, \frac{\Delta_1 - h_1}{2}\right) &= F\left(\frac{\Delta' + 3\Delta - 2h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right), \\ F\left(\frac{\Delta_1' + 3\Delta_1 - 2h_1}{4}, \frac{\Delta_1' - 3\Delta_1}{4}, \frac{\Delta_1 - h_1}{2}\right) &= F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right),\end{aligned}$$

so that the preceding equation gives

$$\sigma = -\sigma,$$

whence $\sigma = 0$. This important conclusion shows that

$$S_1 + S_2 = \sum \left\{ F\left(h, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right) - F\left(\frac{\Delta' + 3\Delta - 2h}{4}, \frac{\Delta' - 3\Delta}{4}, \frac{\Delta - h}{2}\right)\right\},$$

where the sum extends over solutions of (b) subject to the conditions

$$0 < 3\Delta < \Delta', \quad \Delta' + \Delta \equiv 0 \pmod 4, \quad \Delta' - \Delta + 2h \neq 0.$$

But $\Delta' - \Delta + 2h$ can be zero only if n is a perfect square, say s^2, and the corresponding part of the sum is in this case

$$-\sum_{j=0}^{s-1} \{F(2s, s-j, s) - F(2s-j, s-j, s)\}.$$

On the other hand $S_3 = 0$ because of the property $F(x, 0, z) = 0$. Both these remarks, combined with the above established non-trivial equation, lead to the formula (I).

4. *Various Applications.* By selecting the highly arbitrary function $F(x, y, z)$ in a proper way we can obtain a great many more or less interesting results implicitly contained in the above established formulas. Here we confine ourselves to a few consequences of our general formulas bearing upon the problems of representation of numbers by sums of three and five squares. The discussion leading to the final statements in all cases is so easy that without causing any difficulties to the reader we can omit all the details of reasoning.

(a) Let us suppose $n \equiv 3 \pmod 8$ in (III). It is easy to see that the first argument in $F(x, y, z)$ is even throughout and the third argument is odd throughout. We are therefore entitled to take

$$F(x, y, z) = (-1)^{x/2 + (z-1)/2}.$$

The resulting identity if properly interpreted leads to the following conclusion.

If $N_3(n)$ is the number of representations of n as a sum of three squares and

$$g(m) = \sum(-1)^{(d-1)/2}, \quad m = dd', \quad d < d',$$

then

$$N_3(n) = 8[g(n) + 2g(n-2^2) + 2g(n-4^2) + \cdots].$$

This is obviously Liouville's result, which was mentioned in the introduction.

(b) If, supposing $n \equiv 3 \pmod 4$, we take

$$F(x, y, z) = \cos\frac{\pi x}{4} \sin\frac{\pi z}{4}$$

in (III) or

$$F(x, y, z) = \sin\frac{\pi x}{4} \cos\frac{\pi z}{4}$$

in (IV), introduce a new incomplete function

$$G(m) = \sum(-1)^{(d-1)/2+(d^2-1)/8}, \quad m = dd', \quad d < d',$$

and set

$$P = G(n) + 2G(n-4^2) + 2G(n-8^2) + \cdots,$$
$$Q = 2G(n-2^2) + 2G(n-6^2) + 2G(n-10^2) + \cdots,$$
$$R = g(n) - 2g(n-4^2) + 2g(n-8^2) - \cdots,$$

then we obtain the following results:

$$N_3(2n) = 24[P + (-1)^{(n-7)/8}R] \text{ if } n \equiv 7 \pmod 8,$$
$$N_3(2n) = 24[Q + (-1)^{(n-3)/8}R] \text{ if } n \equiv 3 \pmod 8.$$

(c) If, for $n \equiv 5 \pmod 8$, we define $F(x, y, z)$ in (III) as follows:

$$F(x, y, z) = 0, \text{ if } x \text{ is even},$$
$$F(x, 0, z) = 0,$$
$$F(x, y, z) = \cos\frac{\pi x}{4} \sin\frac{\pi z}{4}, \text{ otherwise,}$$

the result obtained is rather curious; namely, setting

$$S = g\left(\frac{n-1^2}{4}\right) + g\left(\frac{n-3^2}{4}\right) + g\left(\frac{n-5^2}{4}\right) + \cdots,$$
$$T = G(n) - 2G(n-4^2) + 2G(n-8^2) - \cdots,$$

we have

$$N_3(n) = 12[2S + (-1)^{(n-5)/8}T] + 12\sum(-1)^{(i-1)/2},$$

where the last sum is extended over all the solutions of the equation

$$n = i^2 + 4j^2$$

in positive numbers.

(d) If, for $n \equiv 1 \pmod 8$, we suppose in (III) and (IV) that $F(x, y, z)$ vanishes whenever y is even, but otherwise is respectively

$$F(x, y, z) = \cos\frac{\pi x}{4}\sin\frac{\pi z}{4},$$

and

$$F(x, y, z) = \sin\frac{\pi x}{4}\cos\frac{\pi z}{4},$$

and take the difference of the results thus obtained, the conclusion is as follows. Using the notation

$$h(m) = \sum(-1)^{(d-1)/2},\ m = dd',\ d < d',\ d \text{ odd},$$
$$h(0) = 0,$$
$$H(m) = \sum(-2/(d' - d)),\ m = dd',\ d < d',\ d \text{ odd},$$
$$H(0) = \tfrac{1}{2},$$

and

$$L = H\left(\frac{n-1^2}{4}\right) - H\left(\frac{n-3^2}{4}\right) - H\left(\frac{n-5^2}{4}\right) + H\left(\frac{n-7^2}{4}\right) + H\left(\frac{n-9^2}{4}\right) - \cdots,$$

$$M = h\left(\frac{n-1^2}{4}\right) + h\left(\frac{n-3^2}{4}\right) + h\left(\frac{n-5^2}{4}\right) + \cdots,$$

we have

$$N_3(n) = 12(L + 2M).$$

(e) A curious result can be obtained from (III) if, supposing $n \equiv 3 \pmod 8$, we take

$$F(x, y, z) = (-1)^{x/2}z.$$

Denoting by $\omega(m)$ the sum of divisors which are less than their conjugate, we obtain

$$\omega(n) - 2\omega(n - 2^2) + 2\omega(n - 4^2) - \cdots = \sum(-1)^{(i-1)/2} i,$$

where the sum is extended over all the solutions of the equation

$$n = i^2 + j^2 + k^2$$

in positive odd numbers. A number of similar results can be derived from the same formulas (III) and (IV), but it would be too long to dwell upon them and we prefer to give some applications of (I) and (II).

5. *Applications of Formulas* (I) *and* (II).

(a) Suppose that in (I) the function $F(x, y, z)$ vanishes whenever y is even, but otherwise

$$F(x, y, z) = \cos\frac{\pi x}{4} \sin\frac{\pi z}{4}.$$

Moreover, we assume $n \equiv 1 \pmod 4$. If we introduce three incomplete functions

$$\phi(m) = \sum(-1)^{(d-1)/2},\ m = dd',\ 3d < d',$$

$$\Phi(m) = \sum(-1)^{(d-1)/2+(d^2-1)/8},\ m = dd',\ 3d < d',$$

$$\psi(m) = \sum(-1)^d \left(\frac{-2}{d' + 3d}\right),\ m = dd',\ 3d < d',\ d + d' \text{ odd},$$

$$\psi(0) = \tfrac{1}{2},$$

and set

$$A = \phi(n) + 2\phi(n - 2^2) + 2\phi(n - 4^2) + \cdots,$$

$$B = \Phi(n) - 2\Phi(n - 4^2) + 2\Phi(n - 8^2) - \cdots,$$

$$C = \psi\left(\frac{n - 1^2}{4}\right) - \psi\left(\frac{n - 3^2}{4}\right) - \psi\left(\frac{n - 5^2}{4}\right) + \psi\left(\frac{n - 7^2}{4}\right) + \psi\left(\frac{n - 9^2}{4}\right) - \cdots,$$

we obtain

$$N_3(n) = 12[A + (-1)^{(n-5)/8} B] \text{ if } n \equiv 5 \pmod 8,$$

$$N_2(n) = 12[A + C] \text{ if } n \equiv 1 \pmod 8.$$

There exists an analogous formula for $n \equiv 3 \pmod 8$.

(b) Suppose that, in the formula (II), the function is defined as follows:

$$F(x, y, z) = 0, \text{ whenever } y \text{ is even},$$
$$F(x, y, z) = 0, \text{ whenever } x \text{ or } x/2 \text{ is odd},$$
$$F(x, y, z) = x, \text{ otherwise},$$

and again $n \equiv 1 \pmod 4$. If $n \equiv 5 \pmod 8$ and $\chi(m)$ is an incomplete function defined by

$$\chi(m) = \sum(d' + 3d), \; m = dd', \; 3d < d',$$

then

$$N_5(n) = 14(\chi(n) + 2\chi(n - 2^2) + 2\chi(n - 4^2) + \cdots),$$

where $N_5(n)$ denotes the number of representations of n as the sum of five squares. This noteworthy result is due to Hermite.

In the case $n \equiv 1 \pmod 8$, we make use of the incomplete functions

$$X(m) = \chi(m) - \chi\left(\frac{m}{4}\right), \; m \text{ even},$$

$$\omega(m) = \sum\left(\frac{-1}{d' + 3d}\right), \; m = dd', \; 3d < d', \; d + d' \text{ odd}.$$

If then we set

$$F = X\left(\frac{n - 1^2}{4}\right) + X\left(\frac{n - 3^2}{4}\right) + X\left(\frac{n - 5^2}{4}\right) + \cdots,$$

$$G = \omega\left(\frac{n - 1^2}{4}\right) - 3\omega\left(\frac{n - 3^2}{4}\right) + 5\omega\left(\frac{n - 5^2}{4}\right) - \cdots,$$

extending these series so far as the arguments remain *positive*, we obtain as a final result

$$N_5(n) = 60[F - G] + 10V,$$

where $V = 0$, if n is not a square,

$$V = \tfrac{1}{4}(3s - (-1)^{(s-1)/2})^2 \quad \text{if} \quad n = s^2.$$

Analogous results exist in the case $n \equiv 3 \pmod 4$.

The preceding formulas are not mere curiosities, but have

real practical value. To show this let us determine the number of representations of $n = 1089 = 33^2$ as the sum of five squares. The necessary numbers are given in the table below.

Arguments	$d'+3d$	$X(m)$	$\omega(m)$
272	275	275	−1
270	273, 99, 57, 141, 63	702	2
266	269, 59, 139	467	−1
260	263, 67, 77	407	−1
252	255, 93, 57, 55, 75	535	−1
242	245, 127	372	0
230	233, 61, 121	415	3
216	219, 81, 51	351	−1
200	203, 55, 49	307	−1
182	185, 47, 97	329	1
162	165, 63, 87, 45	360	0
140	143, 43, 47	233	−3
116	119, 41	160	0
90	93, 39, 33, 51	216	0
62	65, 37	102	2
32	35	35	−1

From this table we easily find

$$F = 5266,\ G = 162,\ 60(F - G) = 306240;$$

and as in this case n is a perfect square ($= 33^2$), we have

$$V = 49^2 = 2401,\quad 10V = 24010.$$

The number of representations of 1089 as the sum of five squares is therefore

$$N_5(1089) = 330250.$$

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ON LOCI OF $(r-2)$-SPACES INCIDENT WITH CURVES IN r-SPACE

BY B. C. WONG

Let s_t lines and t curves $C^{m_1}, C^{m_2}, \cdots, C^{m_t}$ of orders $m_1, m_2, \cdots, m_t$ and deficiencies $p_1, p_2, \cdots, p_t$, respectively, be given in general positions in r-space. In this paper, we propose to determine the number, $N_r^{(t)}$, of $(r-2)$-spaces that are incident with the s_t lines and meet C^{m_1} n_1 times, C^{m_2} n_2 times, $\cdots$, C^{m_t} n_t times, where

$$\text{(A)} \qquad s_t + n_1 + n_2 + \cdots + n_t = 2r - 2,$$

and to deduce a few consequences from the formula for this number. The formula which we shall derive is obviously a function of r, n_i, m_i, and p_i, $(i = 1, 2, \cdots, t)$. The derivation of this formula can be accomplished algebraically,* or by Schubert's symbolic calculus,† by the functional method,‡ or by the method of decomposition.‡ In the present work we find it convenient to adopt the method last named as it yields the desired result with the least difficulty.

A curve C^m may be decomposed in various ways into component curves the sum of whose orders is equal to m, but we wish to decompose it completely, that is, into m lines forming a skew polygon Γ of m sides and $Q^{(1)} = m - 1 + p$ vertices where p is the deficiency of C^m. The non-adjacent vertices of Γ arrange themselves in groups each consisting of a certain number, q, of members. Let $Q^{(q)}$ denote the number of such groups. As we shall have frequent use for this number, we record the following which can be easily verified:

$$Q^{(1)} = \binom{m-1}{1} + \binom{p}{1},$$

* Salmon, *Modern Algebra*, 4th ed., Lesson 19.

† Schubert, *Kalkül der Abzählenden Geometrie*, Leipzig, 1879.

‡ Severi, *Riflessioni intorno ai problemi numerativi concernenti le curve algebriche*, Rendiconti Istituto Lombardo, (2), vol. 54 (1921), pp. 243–254.

$$Q^{(2)} = \binom{m-2}{2} + \binom{m-3}{2}\binom{p}{1} + \binom{p}{2},$$

(B) . . . . . .

. . . . . .

$$Q^{(q)} = \sum_{j=0}^{q} \binom{m-q-j}{q-j}\binom{p}{j};$$

$Q^{(0)}$ is to be taken equal to unity.

Now let $t=0$. Then the symbol $N_r^{(0)}$ denotes the number of $(r-2)$-spaces incident with $s_0=2r-2$ general lines in r-space. Since the number of lines incident with $2r-2$ general $(r-2)$-spaces given in r-space is*

$$\frac{(2r-2)!}{r!(r-1)!},$$

we assume by the principle of duality, or we can prove independently, that $N_r^{(0)}$ is equal to this number; that is,

$$N_r^{(0)} = \frac{(2r-2)!}{r!(r-1)!}. \tag{1}$$

Diminishing r by w, we have

$$N_{r-w}^{(0)} = \frac{(2r-2w-2)!}{(r-w)!(r-w-1)!} \tag{1a}$$

for the number of $(r-w-2)$-spaces that meet $2r-2w-2$ general lines in $(r-w)$-space, which is also the number of $(r-2)$-spaces that pass through w general points and meet $2r-2w-2$ general lines in r-space.

We proceed now to determine, for the case $t=1$, the number $N_r^{(1)}$ of $(r-2)$-spaces that meet n^1 times a given curve C^{m_1} and are incident with $s_1=2r-2-n_1$ given lines in r-space. Replace C^{m_1} by an m_1-sided skew polygon Γ_1 with $Q_1^{(1)}=m_1-1+p_1$ vertices. Any m_1 general lines determine n_1 by n_1 with the s_1 given lines

$$\binom{m_1}{n_1} N_r^{(0)}$$

* C. Segre, *Mehrdimensionale Räume*, Encyklopädie der Mathematischen Wissenschaften, III$_2$, 7, p. 814. Also B. C. Wong, *On the loci of the lines incident with k $(r-2)$-spaces in S_r*, this Bulletin, vol. 34, pp. 715–717.

$(r-2)$-spaces all incident with the s_1 given lines and each incident with n_1 of the m_1 lines. But the m_1 lines or sides of Γ_1 have $Q_1^{(1)}$ incidences each on two of the lines. Through each vertex of Γ_1 pass

$$\binom{m_1-2}{n_1-2}N_{r-1}^{(0)}$$

$(r-2)$-spaces all incident with the s_1 given lines and each incident with n_1-2 of the m_1-2 sides on which the vertex does not lie. Therefore the $Q_1^{(1)}$ vertices of Γ_1 determine

$$\binom{m_1-2}{n_1-2}N_r^{(0)}Q_1^{(1)}$$

such $(r-2)$-spaces. As these $(r-2)$-spaces are improper n_1-uple secant $(r-2)$-spaces of the degenerate curve C^{m_1} incident with the s_1 lines, we deduct their number from

$$\binom{m_1}{n_2}N_r^{(0)}.$$

To the result we now add

$$\binom{m_1-4}{n_2-4}N_{r-2}^{(0)}{}^{(1)}q_1,$$

which is the number of $(r-2)$-spaces each passing through a pair of non-adjacent vertices of Γ_1 and meeting the s_1 given lines and n_1-4 of the m_1-4 sides of Γ_1 not passing through the vertices. Continuing in this manner, we find

$$(2)\qquad N_r^{(1)} = \sum_{q_1=0}^{h_1}(-1)^{q_1}\binom{m_1-2q_1}{n_1-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)},$$

where $h_1=n_1/2$ if n_1 is even and $h_1=(n_1-1)/2$ if n_1 is odd. Replacing r by $r-w$, we have

$$(2a)\qquad N_{r-w}^{(1)} = \sum_{q_1=0}^{h_1}(-1)^{q_1}\binom{m_1-2q_1}{n_1-2q_1}N_{r-w-q_1}^{(0)}Q_1^{(q_1)}$$

as the number of $(r-2)$-spaces that pass through w given general points and meet a given curve C^{m_1} n_1 times and also meet $s_1-2w=2r-2w-2-n_1$ given lines in r-space.

Now let $t=2$. Then $s_2=2r-2-n_1-n_2$. To determine the number, $N_r^{(2)}$, of $(r-2)$-spaces that meet two curves C^{m_1}, C^{m_2} respectively n_1, n_2 times and are incident with s_2 given gen-

eral lines, we decompose C^{m_2} into m_2 lines forming an m_2-sided skew polygon Γ_2 with $Q_2^{(1)} = m_2 - 1 + p_2$ vertices. There are

$$\binom{m_2}{n_2} N_r^{(1)}$$

$(r-2)$-spaces incident with any m_2 general lines of r-spaces n_2 at a time which are also incident with the s_2 given lines. From this number we deduct

$$\binom{m_2-2}{n_2-2} N_{r-1} Q_2^{(1)},$$

which is the number of $(r-2)$-spaces all incident with the s_2 given lines and each incident with a vertex of Γ_2 and with n_2-2 of the m_2-2 sides of Γ_2 on which the vertex does not lie. Continuing as in the preceding paragraph, we find

$$N_r^{(2)} = \sum_{q_2=0}^{h_2} (-1)^{q_2} \binom{m_2 - 2q_2}{n_2 - 2q_2} N_{r-q_2}^{(1)} Q_2^{(q_2)},$$

where $h_2 = n_2/2$ if n_2 is even and $h_2 = (n_2-1)/2$ if n_2 is odd. Putting $w = q_2$ in (2a) and substituting the result in the above we have, after simplifying,

$$(3)\quad N_r^{(2)} = \sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} (-1)^{q_1+q_2} \binom{m_1 - 2q_1}{n_1 - 2q_1} \times \binom{m_2 - 2q_2}{n_2 - 2q_2} N_{r-q_1-q_2}^{(0)} Q_1^{(q_1)} Q_2^{(q_2)}.$$

To determine $N_r^{(3)}, N_r^{(4)}, \cdots$, we proceed in a similar manner. Finally, we arrive at the desired formula:

$$(4)\quad N_r^{(t)} = \sum_{q_1=0}^{h_1} \sum_{q_2=0}^{h_2} \cdots \sum_{q_t=0}^{h_t} (-1)^q N_{r-q}^{(0)} \times \binom{m_1 - 2q_1}{n_1 - 2q_1} \binom{m_2 - 2q_2}{n_2 - 2q_2} \cdots \binom{m_t - 2q_t}{n_t - 2q_t} Q_1^{(q_1)} Q_2^{(q_2)} \cdots Q_t^{(q_t)},$$

where

$$q = q_1 + q_2 + \cdots + q_t,$$

and

$$h_i = n_i/2, \text{ if } n_i \text{ is even},$$

and

$$h_i = (n_i - 1)/2, \text{ if } n_i \text{ is odd}.$$

Now we deduce a few consequences from this formula. For

$t=0$ and $t=1$, we have (1) and (2) respectively. If we put in(2) $n_1=2r-2$, we obtain, since $h_1=r-1$,

$$(5) \qquad N_r^{(1)} = \sum_{q_1=0}^{r-1}(-1)^{q_1}\binom{m_1-2q_1}{2r-2-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)},$$

as the number of $(2r-2)$-secant $(r-2)$-spaces of an r-space curve C^{m_1}. For $r=2$ and $r=3$, (5) becomes respectively

$$\overline{N}_2^{(1)} = \binom{m_1}{2} - Q_1^{(1)} = \frac{1}{2}(m_1-1)(m_1-2) - p_1,$$

and

$$\begin{aligned}\overline{N}_3^{(1)} &= \sum_{q_1=0}^{2}(-1)^{q_1}\binom{m_1-2q_1}{4-2q_1}N_{3-q_1}^{(0)}Q_1^{(q_1)} \\ &= \frac{1}{12}(m_1-2)(m_1-3)^2(m_1-4) - \frac{1}{2}(m_1-3)(m_1-4)p_1 \\ &\quad + \frac{1}{2}p_1(p_1-1),\end{aligned}$$

the former giving the number of double points on a plane curve C^{m_1} of deficiency p_1 and the latter giving the number of quadrisecant lines of a 3-space curve C^{m_1} of deficiency p_1.

If we put $m_1=2r-1$, $p_1=0$ in (5), we have, taking account of (B) and (1a),

$$\sum_{q_1=0}^{r-1}(-1)^{q_1}\frac{(2r-q_1-1)!}{q_1!(r-q_1)!(r-q_1-1)!},$$

which is equal to unity. That is, a rational curve C^{2r-1} of order $2r-1$ in r-space has one and only one $(2r-2)$-secant $(r-2)$-space.

Again, if we put in (2) $n_1=2r-3$ and hence $s_1=1$, $h_1=r-2$, we obtain

$$(6) \qquad \overline{N}_r^{(1)} = \sum_{q_1=0}^{r-2}(-1)^{q_1}\binom{m_1-2q_1}{2r-3-2q_1}N_{r-q_1}^{(0)}Q_1^{(q_1)}.$$

This is the number of $(2r-3)$-secant $(r-2)$-spaces of an r-space curve C^{m_1} that meet a given line, and is therefore the order of the hypersurface formed by the ∞^1 $(2r-2)$-secant $(r-2)$-spaces of C^{m_1}. For $r=2$, the formula gives m_1, that is, the locus of points on a plane curve C^{m_1} is the curve itself. For $r=3$, we have

$$\overline{\overline{N}}_3^{(1)} = \binom{m_1}{3} - \binom{m_1 - 2}{1}(n_{-1} - 1 + p_1)$$

$$= \frac{1}{3}(m_1 - 1)(m_1 - 2)(m_1 - 3) - (m_1 - 2)p_1,$$

for the order of the trisecant surface of a 3-space curve C^{m_1}.

It is of interest to note that the result of substituting $m_1 = 2r-2$ and $p_1 = 0$ in (6) is, if account be taken of (B) and (1a),

$$\sum_{q_1=0}^{r-2}(-1)^{q_1}\frac{(2r - q_1 - 2)!(2r - 2q_1 - 2)}{q_1!(r - q_1)!(r - q_1 - 1)!} = 2.$$

Therefore, the locus of the ∞^1 $(r-2)$-spaces that meet a rational r-space curve C^{2r-2} of order $2r-2$ is always a quadric hypersurface.

Returning to the general formula (4), we see that it is identical with (3) if $t=2$. Let $s_2=0$. Then, from (A), $n_1+n_2=2r-2$. Consider the case $n_1=n_2=r-1$. Then (3) becomes

$$(7)\quad N_r^{(2)} = \sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_1}(-1)^{q_1+q_2}\binom{m_1 - 2q_1}{r - 1 - 2q_1} \times \binom{m_2 - 2q_2}{r - 1 - 2q_2} N_{r-q_1-q_2}^{(0)} Q_1^{(q_1)} Q_2^{(q_2)},$$

where $h_1 = h_2 = (r-1)/2$ if r is odd and $h_1 = h_2 = (r-2)/2$ if r is even. This gives the number of common $(r-1)$-secant $(r-2)$ spaces of two curves C^{m_1} and C^{m_2} in r-space. The case $m_1 = m_2 = r$ and $p_1 = p_2 = 0$ is worth noting. Formula (7) for this case gives

$$\sum_{q_1=0}^{h_1}\sum_{q_2=0}^{h_2}(-1)^{q_1+q_2}\frac{(r - 2q_1)(r - 2q_2)}{(r - q_1 - q_2)} \times \binom{r - q_1}{q_1}\binom{r - q_2}{q_2}\binom{2r - 2q_1 - 2q_2 - 2}{r - q_1 - q_2 - 1}$$

$$= \sum_{j=0}^{k}(r - 2j)^2, \qquad \left[\begin{array}{l} k = r/2 \text{ if } r \text{ is even and} \\ k = (r-1)/2 \text{ if } r \text{ is odd} \end{array}\right]$$

$$= \frac{1}{6}r(r + 1)(r + 2)$$

as the number of common $(r-1)$-secant $(r-2)$-spaces of two normal curves of order r in r-space.* Thus, two twisted cubic curves in 3-space have 10 common secant lines.

As another application of the general formula (4) we give the following. Let

$$n_1+n_2+\cdots+n_t=2r-4.$$

Hence, from (A), $s_t=2$. Then formula (4) gives the number of $(r-2)$-spaces that are incident with two given lines and meet t curves $C^{m_i}\,n_i$ times where $\sum_{i=1}^{t} n_i=2r-4$. This is also the order of the hypersurface V_{r-1} formed by the ∞^1 $(r-2)$-spaces that are incident with a given line and meet $C^{m_i}\,n_i$ times. The ∞^2 $(r-2)$-spaces incident with $C^{m_i}\,n_i$ times meet a general 3-space, and in particular, a 3-space passing through l, in the lines of a congruence K the sum of whose order μ and class ν is the order of the hypersurface V_{r-1}. The order of K is evidently $N_{r-1}^{(t)}$, obtained from (4) by changing r to $r-1$, for this is the number of $(r-2)$-spaces that pass through a given point and meet $C^{m_i}\,n_i$ times. Therefore, the class of K or the number of $(r-2)$-spaces that meet $C^{m_i}\,n_i$ times and meet a given plane in lines is

$$\nu = N_r^{(t)} - N_{r-1}^{(t)}.$$

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* This result can also be obtained from (5) by putting $m_1=2r$ and $p_1=-1$.

THE IDENTICAL EQUATIONS OF THE MULTIPLICATIVE FUNCTION

BY R. VAIDYANATHASWAMY

1. *Introduction.* An arithmetic function $f(N)$ is *multiplicative*, if $f(MN)=f(M)f(N)$, whenever the integers M, N are relatively prime. An analogous definition may be given for functions of more than one argument; thus $\psi(M_1, M_2)$ would be said to be *multiplicative*, if $\psi(M_1N_1, M_2N_2)=\psi(M_1, M_2)\psi(N_1, N_2)$, whenever the two products M_1M_2, N_1N_2 are relatively prime. If $f(N)$ and $\psi(M_1, M_2)$ are multiplicative functions, it is clear that we have

$$f(1) = \psi(1, 1) = 1. \tag{1}$$

If the relation $f(MN)=f(M)f(N)$ is true for *all* integers M, N, we shall say that f is a *linear function.*

The process of *composing* two arithmetic functions $f(N), \phi(N)$ consists in forming their *composite* (represented by $f\cdot\phi$), namely the function ψ defined by the equation

$$\psi(N) = \sum f(\delta)\phi(N/\delta),$$

where the summation is for all divisors δ of N. If f and ϕ are multiplicative, it is easy to see that their composite ψ is also multiplicative. The notion of composition may be obviously extended to functions of several arguments.

Given an arithmetic function $f(N)$, such that $f(1)\neq 0$, it has been shown by E. T. Bell,* that there exists a unique arithmetic function $\psi(N)$, such that the composite $f\cdot\psi$ vanishes for all values of its argument other than 1, and takes the value 1 when the argument is equal to 1. We call ψ the *inverse function* of f, and denote it by f^{-1}; it is easy to see that f^{-1} is multiplicative if f is also.

We shall say that a multiplicative function $F(M_1, M_2)$ is a *cardinal function* of M_1, M_2, if it vanishes whenever either of

* *On a certain inversion in the theory of numbers*, Tôhoku Mathematical Journal, vol. 17 (1920). Also *A ray of numerical functions*, this Bulletin, vol. 32 (1926), p. 341.

M_1, M_2 admits a divisor relatively prime to the other, in other words, whenever the distinct prime factors of M_1 are not identical with those of M_2. We require in what follows, a particular cardinal function $C(M_1, M_2)$ defined as follows:

$C(M_1, M_2) = 0$, if the distinct prime factors of M_1, M_2 are not identical,

$C(M_1, M_2) = (-1)^\nu$, if M_1, M_2 have the same ν distinct prime factors.

I have shown elsewhere,* by the method of generating series, that every multiplicative function f (of a single argument) satisfies a certain identical equation. By using the cardinal function C, the identical equation can be put into the form

$$f(MN) = \sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2) C(\delta_1, \delta_2),$$

the summation on the right extending over all divisors δ_1 of M, and δ_2 of N. My purpose in the present note is to show that this identical equation can be derived, by straightforward reasoning of a purely arithmetical nature, from certain fundamental properties of the inverse function.

2. *Three Theorems relating to the Inverse Function.*

THEOREM 1. *If f is any multiplicative function of one argument, and f^{-1} its inverse function, then the sum*

$$\sum f(M_1\delta) f^{-1}\left(\frac{M_2 N}{\delta}\right),$$

extended over all the divisors δ of N, vanishes unless every prime factor of N divides M_1M_2.

PROOF. Let $N = N_1N_2$, where all the prime factors of N_1 divide M_1M_2, and N_2 is relatively prime to M_1M_2. It is clear that N_1 and N_2 are relatively prime, and therefore any factor δ of N can be expressed uniquely in the form $\delta_1\delta_2$, where δ_1 is a divisor of N_1, and δ_2 of N_2. Hence we have

$$\sum f(M_1\delta) f^{-1}\left(\frac{M_2 N}{\delta}\right) = \sum f(M_1\delta_1\delta_2) f^{-1}\left(\frac{M_2 N_1}{\delta_1} \cdot \frac{N_2}{\delta_2}\right)$$

* In a paper on *The theory of multiplicative arithmetic functions*, to appear shortly in the Transactions of this Society.

$$= \left\{ \sum f(\delta_2) f^{-1}\left(\frac{N_2}{\delta_2}\right)\right\}\left\{ \sum f(M_1\delta_1) f^{-1}\left(\frac{M_2N_1}{\delta_1}\right)\right\}$$

$$= 0, \text{ if } N_2 \neq 1,$$

which proves the theorem.

COROLLARY. A factor N_1 of N may be called a *block factor*, if it is relatively prime to N/N_1. The sum

$$\sum f\left(\frac{N}{\delta}\right) f^{-1}(\delta),$$

extended over all the divisors δ of a block factor $N_1(\neq 1)$ of N, vanishes. This may be proved directly or as a consequence of the theorem. The defining property of the inverse function is the particular case of this result which arises when the block factor N_1 coincides with N.

Suppose now that N contains ν distinct prime factors, and $N_i^1, N_i^2, \cdots, N_i^k (k = \binom{\nu}{i})$ are the distinct block factors of N which contain exactly i of the prime factors. Consider the sum

$$A = \sum_N f\left(\frac{N}{\delta}\right) f^{-1}(\delta) - \sum_{k=1}^{\nu} \left\{ \sum_{N^k} f\left(\frac{N}{\delta}\right) f^{-1}(\delta)\right\}$$
$$+ \sum_{k=1}^{\nu(\nu-1)/2} \left\{ \sum_{N^k_{\nu-1}} f\left(\frac{N}{\delta}\right)^{\nu-1} f^{-1}(\delta)\right\} - \cdots$$
$$+ (-1)^{\nu-1} \sum_{k=1}^{\nu} \left\{ \sum_{N_1^k} f\left(\frac{N}{\delta}\right) f^{-1}(\delta)\right\},$$

where N_i^j below $\sum$ indicates that the sum is extended over all the divisors δ of N_i^j. We shall evaluate the expression A in two ways. In the first place, every partial sum in A, except the first, vanishes by the Corollary to Theorem I. Hence we have

$$A = \sum_N f\left(\frac{N}{\delta}\right) f^{-1}(\delta) = 0, \qquad (N > 1).$$

On the other hand consider a particular divisor d of N, containing i distinct prime factors. The coefficient of $f(N/d)f^{-1}(d)$ in A is

$$1 - \binom{\nu - i}{1} + \binom{\nu - i}{2} - \cdots = 0,$$

if $0<i<\nu$, but $=1$ if $i=\nu$. If $d=1$, the coefficient of

$$f(N/1)f^{-1}(1)=f(N)$$

is

$$1-\binom{\nu}{1}+\binom{\nu}{2}-\cdots+(-1)^{\nu+1}\binom{\nu}{\nu-1}=(-1)^{\nu-1};$$

hence we have

$$A=\sum f\left(\frac{N}{t}\right)f^{-1}(t)+(-1)^{\nu-1}f(N),$$

where the summation is for the divisors t of N which contain *all* its distinct prime factors. Since it has been shown that $A=0$, we have the following theorem.

THEOREM 2. *If N contains ν distinct prime factors, then*

$$\sum f\left(\frac{N}{t}\right)f^{-1}(t)=(-1)^{\nu}f(N),$$

where the summation extends over all the divisors t of N which contain all its ν distinct prime factors.

The next theorem is concerned with two numbers M, N which contain the *same* ν distinct prime factors. We shall denote by M_i^k, N_i^k $(k=1, 2, \cdots, \binom{\nu}{i})$ two block factors of M, N respectively, which contain the *same* i prime factors. We shall also write

$$M=M_i^k\overline{M}_i^k, \quad N=N_i^k\overline{N}_i^k,$$

so that $\overline{M}_i^k$, $\overline{N}_i^k$ are respectively prime to M_i^k, N_i^k and are block factors of M, N, containing the *same* $\nu-i$ prime factors. Consider the expression

$$\begin{aligned}B=&\sum f\left(\frac{MN}{\delta}\right)f^{-1}(\delta)\\&+\sum_{k=1}^{\nu}\left\{\sum\sum f\left(\frac{M_1^k}{t}\cdot\frac{\overline{M}_1^k\overline{N}_1^k}{\delta}\right)f^{-1}(N_1^k t\delta)\right\}\\&-\sum_{k=1}^{\nu(\nu-1)/2}\left\{\sum\sum f\left(\frac{M_2^k}{t}\cdot\frac{\overline{M}_2^k\overline{N}_2^k}{\delta}\right)f^{-1}(N_2^k t\delta)\right\}+\cdots\\&-\cdots+(-1)^{\nu-1}\sum f\left(\frac{M}{t}\right)f^{-1}(Nt).\end{aligned}$$

Here the first term of B is a summation over all divisors δ of N. The second (and later) terms contain three summations; the two inner summations relate respectively to *all* divisors δ of $\overline{M}_i^k\overline{N}_i^k$, and to all such divisors t of M_i^k as contain all its distinct prime factors; the outer summation relates to all possible resolutions of M, N into corresponding block factors containing i and $\nu-i$ primes. The signs of the $\nu+1$ terms in B alternate from the second term onwards. In the last term $i=\nu$, and so the outer summation as well as the summation relating to δ, has disappeared, leaving only the summation over all factors t of M containing all its ν prime factors.

As in the proof of Theorem 2, we shall evaluate B in two different ways. In the first place *all terms of B, except the first and the last, vanish.* To see this, take a fixed divisor t of M_i^k (containing all its prime factors), and consider the sum

$$\sum f\left(\frac{M_i^k}{t}\cdot\frac{\overline{M}_i^k\overline{N}_i^k}{\delta}\right)f^{-1}(N_i^k t\delta),$$

extended over all the divisors δ of $\overline{M}_i^k\overline{N}_i^k$. Since $\overline{M}_i^k\overline{N}_i^k$ has no prime factors in common with $M_i^k/t\times N_i^k t=M_i^kN_i^k$, it follows from Theorem 1 that this (and similarly every other sum of the same type) vanishes. It thus follows that

$$B=\sum f\left(\frac{MN}{\delta}\right)f^{-1}(\delta)+(-1)^{\nu-1}\sum f\left(\frac{M}{t}\right)f(Nt),$$

where the first summation is for all divisors δ of N, and the second for all divisors t of M which contain all its ν distinct prime factors.

Secondly, we shall show that B is equal to the sum $\sum f(MN/\delta)f^{-1}(\delta)$, extended over all divisors δ of MN, and is therefore identically zero. A divisor d of MN may divide N; if it does not divide N, then a certain number, say $i(\geqq 1)$, of the ν prime factors of N must occur in d to a higher power than in N. Denote by N_i^k the block factor of N corresponding to these i primes; then we can express d in the form

$$d=N_i^k t\delta,$$

where δ is an arbitrary factor of $\overline{M}_i^k\overline{N}_i^k$, and t is a factor of M_i^k containing all its prime factors (M_i^k, $\overline{M}_i^k$, $\overline{N}_i^k$ having the mean-

ings already explained). But this expression for d is not unique, since there is no restriction on the factor δ of $\overline{M}_i^k\overline{N}_i^k$; in fact, if N_j^l $(1 \leqq j < i)$ is a block factor of N, which divides N_i^k, we can also write

$$d = N_j^l t_1 \delta_1,$$

where t_1 is a divisor of M_j^l containing all its prime factors, and δ_1 is some divisor of $\overline{M}_j^l\overline{N}_j^l$.

Consider now the number of times which $f(MN/d)f^{-1}(d)$ (d being a particular divisor of MN) occurs in (B). In case d divides N, it is clear that it occurs only once, namely in the first term of B. If d does not divide N, let exactly i of the ν prime factors occur in d to a higher power than in N; from the previous explanation, it will be clear then that $f(MN/d)f^{-1}(d)$ will occur $\binom{i}{1}$ times in the second term of B, and generally $\binom{i}{\lambda}$ times in the $(\lambda+1)$th term. Hence its coefficient in B is

$$\binom{i}{1} - \binom{i}{2} + \cdots = 1 - (1-1)^i = 1, \text{ if } i > 0.$$

Thus every term $f(MN/d)f^{-1}(d)$ occurs just once in B, whether d divides N or not. Hence

$$B = \sum_{\delta \mid MN} f\left(\frac{MN}{\delta}\right) f^{-1}(\delta) = 0 \quad (\text{since } MN > 1).$$

Combining this with our previous evaluation of B, we have the following result.

THEOREM 3. *If M and N have the same ν distinct prime factors, then*

$$\sum f\left(\frac{MN}{\delta}\right) f^{-1}(\delta) = (-1)^\nu \sum f\left(\frac{M}{t}\right) f^{-1}(Nt),$$

where on the left the summation is for all divisors δ of N, and on the right for all divisors t of M which contain all its ν distinct prime factors.

COROLLARY 1. *On putting $N=1$ in Theorem 3, we obtain Theorem 2. Hence the theorem must be considered to be true for any M, and $N=1$.*

COROLLARY 2. *Let M_1 be relatively prime to M (and therefore*

also to N). Write $M'=M_1M$, and multiply both sides of the equality in Theorem 3 by $f(M_1)$. We obtain the relation

$$\sum f\left(\frac{M'N}{\delta}\right)f^{-1}(\delta) = (-1)^{\nu}\sum f\left(\frac{M'}{t}\right)f^{-1}(Nt),$$

where M' is any number containing all the ν distinct prime factors of N, the summation on the left is for all divisors δ of N, and on the right for all divisors t of M' which contain only the ν distinct prime factors of N.

COROLLARY 3. *Theorem 3 is also true for any two numbers M, N, provided* (1) *ν is the number of distinct prime factors in N, and* (2) *the summation on the right is for all divisors t of M which contain the ν distinct primes dividing N, and no others.*

If all the ν prime factors of N occur in M, then Corollary 3 reduces to Corollary 2. If on the other hand, some of the ν prime factors of N do not occur in M, then the left side of the equality of Theorem 3 vanishes by virtue of Theorem I, and the right side also vanishes, since there are now no factors t of M which contain *all* the prime factors of N. Thus, under this interpretation of the summation on the right, Theorem 3 is true for any two numbers M, N.

3. *The Identical Equation of f.* Replacing N by N_1, the equality of Theorem 3 is

$$\sum f\left(\frac{MN_1}{\delta}\right)f^{-1}(\delta) = (-1)^{\nu}\sum f\left(\frac{M}{t}\right)f^{-1}(N_1t),$$

where ν is the number of prime factors in N_1, δ runs over the divisors of N_1, and t over those divisors of M which contain neither more nor less than the ν prime factors of N_1. As has been remarked, this result is true for *any* two numbers M, N_1.

Multiply both sides of the equality by $f(N_2)$ and sum over all values of N_1, N_2, such that $N_1N_2=N$. On the left side, we carry out the summation in two stages; namely, we first keep N_1/δ fixed, and sum over all values of N_2 and δ such that $N_2\delta = N\delta/N_1$. Thus the left side is

$$\sum\sum f\left(\frac{MN_1}{\delta}\right)f^{-1}(\delta)f(N_2) = \sum f\left(\frac{MN_1}{\delta}\right)\sum f^{-1}(\delta)f(N_2).$$

We have also

$$\sum_{N_2\delta=N\delta/N_1} f^{-1}(\delta)f(N_2) = 0, \text{ if } \frac{N_1}{\delta} \neq N;$$

$$= 1, \text{ if } \frac{N_1}{\delta} = N.$$

Thus the left side reduces to $f(MN)$. The right side is

$$\sum\sum(-1)^{\nu} f\left(\frac{M}{t}\right) f\left(\frac{N}{\delta}\right) f^{-1}(\delta t),$$

summed for all divisors δ of N, and all such divisors t of M as contain neither more nor less than the ν prime factors of δ. It is clear that this is identical with

$$\sum\sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2) C(\delta_1, \delta_2),$$

δ being the special cardinal function defined above, and the summation extending over *all* the divisors δ_1 of M, and δ_2 of N. The identical equation of f, namely,

$$f(MN) = \sum\sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) f^{-1}(\delta_1\delta_2) C(\delta_1, \delta_2),$$

is thus established.

4. *The Identical Equation of the Quadratic Function.* A multiplicative function f may be called *an integral quadratic function*, or simply *a quadratic function*, if it is the composite of two linear functions. More generally, f is an *integral function of the rth degree*, if it is the composite of r linear functions. We shall now show that the identical equation of f assumes a simple form when it is a quadratic function.

A fundamental property of the inverse function is: *the inverse of the composite of any number of multiplicative functions of one argument is the composite of their inverses.* This is easily proved, and is assumed in what follows.

THEOREM 4. *If $f(N)$ is an integral function of the rth degree, its inverse function $f^{-1}(N)$ vanishes whenever N is divisible by an $(r+1)$th power; also, if N is the product of distinct primes,*

$$f^{-1}(N^r) = \{\mu(N)\}^r \lambda_1(N)\lambda_2(N)\cdots\lambda_r(N),$$

where f is the composite of the linear functions $\lambda_1, \lambda_2, \cdots, \lambda_r$, and $\mu(N)$ is Mertens' function.

For consider the theorem for the case $r=1$, that is, when f is a linear function $\lambda(N)$. If p is an arbitrary prime, we have from the definition of the inverse function

$$\lambda(p) + \lambda^{-1}(p) = 0; \quad \lambda(p^2) + \lambda(p)\lambda^{-1}(p) + \lambda^{-1}(p^2) = 0.$$

Hence $\lambda^{-1}(p) = -\lambda(p)$, and therefore $\lambda^{-1}(N) = \mu(N)\lambda(N)$, when N is a product of distinct primes. Also, remembering that λ is linear, we have $\lambda(p^2) = \lambda(p)\lambda(p)$, and therefore, from the second equation, $\lambda^{-1}(p^2) = 0$. Hence $\lambda^{-1}(N) = 0$ when N has a square factor. The theorem is thus true for $r=1$.

Assume now the truth of the theorem for the value $r-1$ of r, and let f be the composite of the linear functions $\lambda_1, \lambda_2, \cdots, \lambda_r$. Then $f^{-1}\lambda_r$ is the composite of the $r-1$ linear functions $\lambda_1, \lambda_2, \cdots, \lambda_{r-1}$. Hence, by virtue of our assumption, we may write

$$\begin{aligned}
0 = (f^{-1}\cdot\lambda_r)(p^r) &= f^{-1}(p^r) + f^{-1}(p^{r-1})\lambda_r(p) + f^{-1}(p^{r-2})\lambda_r(p^2) + \cdots, \\
0 = (f^{-1}\cdot\lambda_r)(p^{r+1}) &= f^{-1}(p^{r+1}) + f^{-1}(p^r)\lambda_r(p) + \cdots \\
&= f^{-1}(p^{r+1}) + \lambda_r(p)\{f^{-1}(p^r) \\
&\quad + f^{-1}(p^{r-1})\lambda_r(p) + \cdots\}, \\
&= f^{-1}(p^{r+1}),
\end{aligned}$$

since λ_r is linear. Hence $f^{-1}(N) = 0$ if N is divisible by an $(r+1)$th power. Again, for the second part of the theorem, we have, by virtue of our assumption,

$$\begin{aligned}
(-1)^{r-1}\lambda_1(p)\lambda_2(p)\cdots\lambda_{r-1}(p) &= (f^{-1}\cdot\lambda_r)(p^{r-1}) \\
&= f^{-1}(p^{r-1}) + f^{-1}(p^{r-2})\lambda_r(p) + \cdots.
\end{aligned}$$

Hence

$$\begin{aligned}
0 &= (f^{-1}\cdot\lambda_r)(p^r) \\
&= f^{-1}(p^r) + \lambda_r(p)\{f^{-1}(p^{r-1}) + \lambda_r(p)f^{-1}(p^{r-2}) + \cdots\} \\
&= f^{-1}(p^r) + (-1)^{r-1}\lambda_1(p)\lambda_2(p)\cdots\lambda_r(p),
\end{aligned}$$

or

$$\begin{aligned}
f^{-1}(p^r) &= (-1)^r\lambda_1(p)\lambda_2(p)\cdots\lambda_r(p); \\
f^{-1}(N^r) &= \{\mu(N)\}^r\lambda_1(N)\cdots\lambda_r(N),
\end{aligned}$$

N being a product of distinct primes. Thus the theorem is true for the next higher value of r. Since it has been seen to be true for $r=1$, the induction is complete.

THEOREM 5. *If f is the composite of the two linear functions $\lambda_1(N)$, $\lambda_2(N)$, its identical equation takes the form*

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \lambda_1(\delta)\lambda_2(\delta)\nu(\delta),$$

where the summation is now for all common divisors δ of M, N.

PROOF. The identical equation of f may be written in the form

$$f(MN) = \sum f\left(\frac{M}{\delta_1}\right) f\left(\frac{N}{\delta_2}\right) \cdot (-1)^{\nu} f^{-1}(\delta_1\delta_2),$$

summed for pairs of divisors δ_1, δ_2 of M, N, which contain the same distinct prime factors, ν in number. Now if f is a quadratic function, $f^{-1}(\delta_1\delta_2)$ vanishes when $\delta_1\delta_2$ contains a cubed factor and therefore (since δ_1, δ_2 contain the same distinct prime factors) when either δ_1 or δ_2 contains a squared factor. It follows that the only non-zero terms in the above sum are those for which

$$\delta_1 = \delta_2 = \delta = \text{a product of distinct primes.}$$

Hence

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \cdot (-1)^{\nu} f^{-1}(\delta^2),$$

summed for common divisors δ of M, N. If we use Theorem 4, this becomes

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) f^{-1}(\delta^2)\mu(\delta)$$

$$= \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \lambda_1(\delta)\lambda_2(\delta)\mu(\delta).$$

As special cases of this theorem, may be mentioned the identical equations of the following functions:

(1) $\sigma_a(N)$ (= the sum of the ath powers of the divisors of N). This is a quadratic function, being the composite of the two linear functions $\lambda_1(N) = N^a$, and $\lambda_2(N) = 1$. Hence

$$\sigma_a(MN) = \sum \sigma_a\left(\frac{M}{\delta}\right)\sigma_a\left(\frac{N}{\delta}\right)\delta^a\mu(\delta),$$

summed for common divisors δ.

(2) The function $4R(N)$ which is equal to the number of representations of N as a sum of two squares. It is known that $R(N)$ is also equal to the excess of the number of divisors of N of the form $4n+1$ over the number of those of the form $4n-1$. Hence $R(N)$ is a quadratic function, being the composite of the linear functions $\lambda_1(N)$, $\lambda_2(N)$ defined by the quadratic residue symbol

$$\lambda_2(N) = 1;\ \lambda_1(2^n) = 0;\ \lambda_1(N) = \left(\frac{-1}{N}\right),$$

Hence

$$R(MN) = \sum R\left(\frac{M}{\delta}\right)R\left(\frac{N}{\delta}\right)\left(\frac{-1}{\delta}\right)\mu(\delta),$$

summed for common divisors δ of M, N.

MADRAS UNIVERSITY,
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NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ALLIAUME (M.) et DORY (M.). Traité de trigonométrie. Namur, Wesmael-Charlier, 1930. 8+398 pp.

DORY (E.). See ALLIAUME (M.).

DURAND (G.). Pour comprendre la trigonométrie. Paris, Doin, 1930. 250 pp.

FORT (T.). Infinite series. Oxford, Clarendon Press, 1930. 4+253 pp.

FRANKE (W.). Mathematische Formelsammlung. Braunschweig, Vieweg, 1930.

GARNIER (R.). Cours de mathématiques générales (analyse et géométrie). Tome I: Calcul différentiel. Géométrie. Paris, Gauthier-Villars, 1930. 6+463 pp.

HADAMARD (J.). Cours d'analyse. Tome 2: Potentiel; calcul des variations; fonctions analytiques; équations différentielles et aux dérivées partielles; calcul des probabilités. Paris, Hermann, 1930. 6+721 pp.

HASSLER (J. O.) and SMITH (R. R.). The teaching of secondary mathematics. Edited by E. R. Hedrick. New York, Macmillan, 1930. 12+405 pp.

HAWKINGS (J.). Superficial measurement. London, Crosby Lockwood, 1930. 172 pp.

HEDRICK (E. R.). See HASSLER (J. O.).

ILLSUNG (F.). Die Descente-Methode für die allgemeine Lösung des Fermat-Problems. Würzburg, C. J. Becker, Universitätsdruckerei, 1930. 16 pp.

VON JUHOS (B.). Das Problem der mathematischen Wahrscheinlichkeit. München, Ernst Reinhardt, 1930. 93 pp.

LORIA (G.). Curve piane speciali algebriche e trascendenti. Teoria e storia. Volume II: Curve trascendenti, curve dedotte da altre. Milano, Hoepli, 1930. 12+439 pp.

LYCHE (R. T.). Begynnelsesgrunnene av differensial- og integralregningen. Oslo, Aschehoug, 1930. 61 pp.

RUSSELL (A. H.). Checking calculations. For accountants, clerks and teachers. London and New York, Gregg, 1930. 64 pp.

SAINTE-LAGUË (A.). Géométrie de situation et jeux. (Mémorial des Sciences Mathématiques, No. 41.) Paris, Gauthier-Villars, 1930.

SANFORD (V.). A short history of mathematics. Boston, Houghton-Mifflin, 1930. 12+402 pp.

SCANDINAVIAN MATHEMATICAL CONGRESS. Comptes rendus du Septième Congrès des Mathématiciens Scandinaves tenu à Oslo 19–22 Août 1929. Oslo, Brøggers Boktrykkeri, 1930. 19+197 pp.

SMITH (R. R.). See HASSLER (J. O.).

TITCHMARSH (E. C.). The zeta-function of Riemann. (Cambridge Tracts, No. 26.) Cambridge, University Press, 1930. 6+104 pp.

WYCKOFF (R. W. G.). The analytical expression of the results of the theory of

space groups. 2d edition. Washington, Carnegie Institution, 1930. 239 pp.

PART II. APPLIED MATHEMATICS

ANDRADE (E. N. DA C.). The mechanism of nature, being a simple approach to modern views on the structure of matter and radiation. London, Bell, 1930. 12+170 pp.

BAKER (R. H.). Astronomy. An introduction. New York, Van Nostrand, 1930. 20+521 pp.

BÉRILLE (L.). Tables pour le tracé des courbes. Paris, Béranger, 1929. 40 pp.

BRILLOUIN (L.). Les statistiques quantiques et leurs applications. Tomes I et II. Paris, Les Presses Universitaires de France, 1930. 192+212 pp.

DE BROGLIE (L.). An introduction to the study of wave mechanics. Translated from the French by H. T. Flint. London, Methuen, 1930. 6+249 pp.

——— See DE BROGLIE (M.).

DE BROGLIE (M.) und DE BROGLIE (L.). Einführung in die Physik der Röntgen- und Gamma-Strahlen. Übersetzt von L. von Seuffert. Leipzig, Barth, 1930. 8+208 pp.

BROSE (H. L.). See SOMMERFELD (A.).

BRUNOLD (C.). L'entropie. Son rôle dans le développement historique de la thermodynamique. Paris, Masson, 1930. 211 pp.

——— Le problème de l'affinité chimique et l'atomistique. Paris, Masson, 1930. 118 pp.

BURBACH (T.) und HERMANN (R.). Strömungswiderstand und Wärmeübergang in Rohren. Leipzig, Akademische Verlagsgesellschaft, 1930. 4+88 pp.

CAMPBELL (N. R.) and RITCHIE (D.). Photo-electric cells. Their properties, use and applications. London, Pitman, 1929. 7+209 pp.

CASTELFRANCHI (G.). Fisica moderna. Visione sintetica, pianamente esposta, della fisica d'oggi e dei lavori dei maggiori fisici contemporanei. 2a edizione. Milano, Hoepli, 1930. 10+856 pp.

CHAPMAN (F. T.). A study of the induction motor. London, Chapman and Hall, 1930. 16+289 pp.

CHOISNARD (P.). La méthode statistique et le bon sens en astrologie scientifique. Paris, Alcan, 1930.

COUDERC (P.). L'architecture de l'univers. Paris, Gauthier-Villars, 1930. 176 pp.

CRAWFORD (R. T.). Determination of comets and asteroids. New York, McGraw-Hill, 1930. 233 pp.

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DIRAC (P. A. M.). The principles of quantum mechanics. Oxford, Clarendon Press, 1930. 10+257 pp.

DE DONDER (T.). Applications de la gravifique einsteinienne. (Mémorial des Sciences Mathématiques, No. 43.) Paris, Gauthier-Villars, 1930.

DUBROCA (M.). La doctrine einsteinienne devant les faits périodiques et la réfraction lumineuse. Paris, Gauthier-Villars, 1930. 128 pp.

DUNCAN (W. J.). See FRAZER (R. A.).

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FLEMING (R.). Wind stresses in buildings. New York, Wiley, 1930. 193 pp.

FLINT (H. T.). See DE BROGLIE (L.).

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FRAZER (R. A.) and DUNCAN (W. J.). The flutter of aeroplane wings. London, H. M. Stationery Office, 1929.

GANS (R.). Die Physik der letzten dreissig Jahre. Königsberg, Gräfe und Unger, 1930. 19 pp.

GIAO (A.). La mécanique différentielle des fronts et du champ isallobarique. Paris, Office National Météorologique, 1929.

GOSSELIN (A.) et GOSSELIN (M.). Constitution et thermochimie des molécules. Paris, Les Presses Universitaires de France, 1930. 7+231 pp.

GOSSELIN (M.). See GOSSELIN (A.).

GUTTON (C.). Les ondes électriques de très courtes longueurs et leurs applications. Paris, Hermann, 1930. 20 pp.

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HERMANN (R.). See BURBACH (T.).

HOSMER (G. L.), edited by. Geodesy, including astronomical observations. 2d edition. New York, Wiley, 1930.

HUMPHREY. (D.). Intermediate mechanics. Dynamics. London and New York, Longmans, Green, 1930. 12+382 pp.

JAHR (W.) und KNECHTEL (P.). Grundzüge der Getriebelehre. Band 1. Leipzig, Jänicke, 1930. 15+402 pp.

KLEINLOGEL (A.). Rahmenformeln. 6te vollkommen neubearbeitete und wesentlich erweiterte Auflage. Berlin, Ernst, 1929. 24+455 pp.

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LEHR (E.). Schwingungstechnik. Ein Handbuch für Ingenieure. Band 1: Grundlagen. Die Eigenschwingungen eingliedriger Systeme. Berlin, Springer, 1930. 23+295 pp.

MACKENZIE (A. H.) and FORSTER (A.). Theoretical and practical mechanics and physics, with a chapter on chemistry. London, Macmillan, 1930. 18+288 pp.

MATOSSI (F.). See SCHAEFER (C.).

PÖSCHL (T.). Lehrbuch der technischen Mechanik. 2te vollständig umgearbeitete Auflage. Berlin, Springer, 1930. 8+318 pp.

POHL (R. W.). Einführung in die Mechanik und Akustik. Berlin, Springer, 1930. 8+250 pp.

POORMAN (A. P.). Applied mechanics. 3d edition. New York, McGraw-Hill, 1930. 306 pp.

RITCHIE (D.). See CAMPBELL (N. R.).

ROTHÉ (E.). Les ondes séismiques et leur propagation. (Mémorial des Sciences Physiques, No. 12.) Paris, Gauthier-Villars, 1930.

ROUTH (E. J.). The advanced part of a treatise on the dynamics of a system of rigid bodies. Being Part 2 of a treatise on the whole subject. 6th edition, revised and enlarged. London, Macmillan, 1930. 14+484 pp.

SCHAEFER (C.) und MATOSSI (F.). Der Ramaneffekt. (Fortschritte der Chemie, Physik und Physikalischen Chemie.) Berlin, Borntraeger, 1930. 3+52 pp.

SCHIAPARELLI (G. V.). Le opere di G. V. Schiaparelli. Pubblicate per cura della Reale Specola di Brera. Tomo 2. Milano, Hoepli, 1930. 4+486 pp.

SCHLEICHER (F.). Statik. Teil 1. (Sammlung Göschen.) Berlin, de Gruyter, 1930. 143 pp.

SEELYE (H. P.). Electrical distribution engineering. New York, McGraw-Hill, 1930. 8+710 pp.

VON SEUFFERT (L.). See DE BROGLIE (M.).

SIEBRECHT (W.). Beitrag zur Regelung der Kreiselpumpen und Untersuchungen über die theoretische und wirkliche Förderhöhe. Berlin, Verein der Deutschen Ingenieure, 1929.

SMITH (K. K.). See CREW (H.).

SOMMERFELD (A.). Wave-mechanics. Translated from the German edition by H. L. Brose. Supplementary volume to "Atomic structure and spectral lines." London, Methuen, 1930. 12+304 pp.

SZEGÖ (G.). See WEBSTER (A. G.).

WEBSTER (A. G.). Partielle Differentialgleichungen der mathematischen Physik. Deutsche Bearbeitung von G. Szegö. Leipzig, Teubner, 1930. 8+428 pp.

WEINBERGER (O.). Mathematische Volkswirtschaftslehre. Eine Einführung. Leipzig, Teubner, 1930. 14+241 pp.

WOOD (A. B.). A textbook of sound. Being an account of the physics of vibrations, with special reference to recent theoretical and technical developments. London, Bell, 1930.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, October 25, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than October 4. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Secretary not later than September 7. By invitation of the program committee, Professor M. H. Stone will deliver an address entitled *Group representations in Hilbert space.*

COLUMBIA, MISSOURI, November 28-29, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than November 6. All abstracts received by that date will appear in the Bulletin in advance of the meeting. On Saturday morning, Professor C. N. Moore will deliver an address by invitation of the program committee, on *Types of series and types of summability;* and Professor R. E. Langer will deliver an address on *The zeros of exponential sums and of certain related functions.*

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, November 29, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, Berkeley, California, not later than November 1. All abstracts received by that date will appear in the Bulletin in advance of the meeting. At the request of the program committee, addresses will be delivered by Professor Harald Bohr on *The theory of Dirichlet series,* and by Professor H. F. Blichfeldt on *The method of geometry of numbers.*

CLEVELAND, OHIO, ANNUAL MEETING, December 29, 1930, to January 1, 1931.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than December 1. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than November 7. There will be the usual joint session with Section A of the American Association for the Advancement of Science and with the Mathematical Association of America. There will be invited addresses by Professors G. D. Birkhoff, E. T. Bell, and others. A joint session with the American Statistical Society has been arranged for and it is probable that there will be a joint session with the American Physical Society. Professor E. B. Wilson will deliver the eighth Josiah Willard Gibbs Lecture.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, G. W. MULLINS, 501 West 116th Street, New York.

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